J Geom Anal (2019) 29:121–133
https://doi.org/10.1007/s12220-018-9982-9

Uniqueness of Minimal Graph in General Codimension

Yng-Ing Lee[1,2] · **Yuan Shyong Ooi**[1] · **Mao-Pei Tsui**[1,2]

Received: 12 October 2017 / Published online: 17 January 2018

Abstract In this paper, we obtain the uniqueness of general codimension Dirichlet problem for minimal surface system in restricted classes. The condition is in terms of singular values and in particular covers the classical hypersurface case and earlier results in higher codimension. To prove the uniqueness result, a natural way is to consider the geodesic homotopy of two solutions. However, the singular values for linear combination of maps are not clear. We apply majorization techniques from convex optimisation to overcome the difficulties.

1 Introduction

Let $f = (f^1, \dots, f^m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^2 vector-valued function. We say that f is a solution of the minimal surface system with boundary data ϕ if

$$\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f^\alpha}{\partial x^j}\right) = 0 \quad \text{for each } \alpha = 1, \dots, m, \text{ on } \Omega \\ &f = \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $g_{ij} = \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}$ and g^{ij} is the inverse matrix of (g_{ij}).

✉ Yuan Shyong Ooi
d03221001@ntu.edu.tw

Yng-Ing Lee
yilee@math.ntu.edu.tw

Mao-Pei Tsui
maopei@math.ntu.edu.tw

1 Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

2 National Center for Theoretical Sciences, Mathematics Division, Taipei 10617, Taiwan

Lawson and Osserman studied minimal surface system (1.1) in high codimension in [2]. Unlike in the codimension one case, they showed that higher codimension Dirichlet problem for minimal surface system does not necessarily have existence, uniqueness, stability and regularity results. In particular, Lawson and Osserman constructed counter examples for both stability and uniqueness for the case $n = m = 2$.

Higher codimension stability and uniqueness problems for minimal surface system were studied in [4,5] by Lee and Wang in a general Riemannian setting. In [5], Lee and Wang derived the second variational formula for area functional in terms of singular values of the graph function, and found several criteria for the stability of the minimal surface system in terms of singular values. Specifically, they proved that distance decreasing condition or a condition on the 2-Jacobian will imply the stability of minimal graph. The results included the classical case that every codimension one minimal graph is stable. Moreover, they were able to obtain uniqueness result of minimal graph of distance decreasing map.

Along the same approach, Lee-Tsui [3] further investigated the stability problem by studying convexity of the area functional $\mathcal{A}$, where

$$\mathcal{A}(x_1,\ldots,x_n) = \sqrt{\prod_{i=1}^{n}(1+x_i^2)} \quad \text{on} \quad \mathbb{R}^n_{\geq 0} := \{(x_1,\ldots,x_n) : x_i \geq 0\} \tag{1.2}$$

They showed that

Theorem A [3, Theorem 4.1] *Let M, N be Riemannian manifolds and Σ be the graph of a map $f : M \to N$. Suppose that N has non-positive sectional curvature and Σ is minimal in $M \times N$. Then Σ is stable if*

$$\lambda_i \lambda_j \leq 1 \qquad \text{for } 1 \leq i \neq j \leq n \tag{1.3}$$

and

$$\prod_{i=1}^{n}\left(1-\lambda_i^2\right) + \sum_{i=1}^{n}\left(1-\lambda_1^2\right)\cdots\left(1-\lambda_{i-1}^2\right)\lambda_i^2\left(1-\lambda_{i+1}^2\right)\cdots\left(1-\lambda_n^2\right) \geq 0 \tag{1.4}$$

where $\lambda_i \geq 0$ are the singular values of df.

Note that the conditions of (1.3) and (1.4) are equivalent to the convexity of the area functional $\mathcal{A}$. Theorem A recovered the stability result in [5] for distance decreasing case, and also implied new stability results for the cases

(a) $\prod_{i=1}^{n}(1+\lambda_i^2) \leq 4, \qquad$ or

(b) $|\lambda_i \lambda_j| \leq \dfrac{1}{\sqrt{p-1}}$ for $1 \leq i \neq j \leq n$,

where $p = \operatorname{rank} df$. The case (b) generalises a result in [5].

A natural question to ask is whether these stability results can lead to some uniqueness theorems on the minimal surface system in general codimension. This problem is not addressed in Lee-Tsui's paper and is the main goal of our paper.

Denote $q = \min\{n, m\}$ and $\mathcal{M}$ to be the subset in $\mathbb{R}^q_{\geq 0}$ that satisfies the strict conditions of (1.3) and (1.4), i.e.

$$x_i x_j < 1 \qquad \text{for } 1 \leq i \neq j \leq q \tag{1.5}$$

and

$$\prod_{i=1}^{q}\left(1 - x_i^2\right) + \sum_{i=1}^{q}\left(1 - x_1^2\right)\cdots\left(1 - x_{i-1}^2\right)x_i^2\left(1 - x_{i+1}^2\right)\cdots\left(1 - x_q^2\right) > 0. \tag{1.6}$$

The *strict convexity* of the functional $\mathcal{A}$ is equivalent to the singular values that lie in the set $\mathcal{M}$.

We need a definition before stating , our main theorem.

Definition 1.1 A subset $A \subset \mathbb{R}^n$ is said to be *symmetric* if it satisfies the property: Given any $(x_1, \ldots, x_n) \in A$ we have $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in A$ for any permutation function π of 1 to n.

We prove in this paper the following theorem,

Theorem (Theorem 3.1) *Assume the graphs of $f_0 : \Omega \subset M^n \to \mathbb{R}^m$ and $f_1 : \Omega \subset M^n \to \mathbb{R}^m$ are both minimal submanifolds in $M \times \mathbb{R}^m$ with the same boundary data. If both the singular value vectors of f_0 and f_1 lie in a symmetric convex subset of $\mathcal{M}$, then $f_0 = f_1$.*

An extension of the theorem to non-flat target space is obtained in Theorem 5.2. In Sect. 4, we give several explicit symmetric convex subsets and obtain uniqueness results in these situations. It in particular includes the classical hypersurface case, an optimal result for two dimensional surface or codimensional two case, the class of distance decreasing maps, the class satisfying $\left\{\prod_{i=1}^{n}(1+\lambda_i^2) < 4\right\}$, and some other cases.

To prove this theorem, it is natural to consider the geodesic homotopy of two solutions and show that the area functional is convex along the homotopy. However, the convexity property we have is in terms of singular values of the map and it is not clear whether the geodesic homotopy of two convex critical points of the area functional still satisfies the convex condition. We overcome the difficulties by applying majorization techniques in convex optimisation to find suitable conditions. Note that in [5], they use the convexity of Jacobi field to control the singular values, but the method only works for distance decreasing case. We give a generalisation of the approach in Theorem 5.2.

The proof of the Theorem 3.1 consists of two parts. The first part is to show the convexity along the geodesic homotopy. Two key lemmas in weak majorization by Ky Fan [1] and Mirsky [6], respectively, will play important roles in this paper. Specifically we show that if the singular value vectors of two matrices lie in a certain symmetric convex set, then the singular values vector of their matrices convex combination will lie in a bigger convex set. The second part is on the uniqueness. It involves estimates on the second variational formula of the area functional.

This paper is organised as follows: Sect. 2 will give a brief introduction on singular value decomposition and the concept of majorization. We then state lemma 2.2 and lemma 2.3 that are essential for the Theorem 3.1. Section 3 will be dedicated to prove the main theorem. In Sect. 4, we give some applications of the theorem by giving new explicit examples of convex set for the uniqueness results. The last section will consider uniqueness result in a more general Riemannian manifold.

2 Preliminary

In this section, we recall some notation and formulae that will be used later.

2.1 Singular Value Decomposition

Suppose that (M, g) and (N, h) are two Riemannian manifolds with dimension n and m, respectively. Let f be a smooth map from (M, g) to (N, h), and df be the differential of f. We can define its adjoint $(df)^\top$ via the metric,

$$\langle (df)^\top(v), w\rangle_g := \langle v, df(w)\rangle_h$$

for $w \in TM$, $v \in TN$.

The graph of f is an embedded submanifold Σ in the product manifold $M \times N$. Denote the projections by $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$. For simplicity, we still denote the differentials by $\pi_1 : T_p(M \times N) \to T_{\pi_1(p)}M$ and $\pi_2 : T_p(M \times N) \to T_{\pi_2(p)}N$ at any point $p \in M \times N$.

Let $\{\lambda_i \geq 0\}$ be the eigenvalue of $\sqrt{(df)^\top df}$ which will be referred as singular values of df or f. Fix any point $p \in \Sigma$, and let $r = \operatorname{rank} df(p)$. We can rearrange them so that $\lambda_i = 0$ for $i \geq r + 1$. Then singular value decomposition tells us that there exist orthonormal bases $\{a_i\}_{i=1\ldots n}$ for $T_{\pi_1(p)}M$ and $\{b_i\}_{i=1\ldots m}$ for $T_{\pi_2(p)}N$ such that

$$df(a_i) = \begin{cases} \lambda_i b_i & 1 \leq i \leq r \\ 0 & r < i \leq n. \end{cases}$$

Moreover,

$$e_i = \begin{cases} \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i b_i) & 1 \leq i \leq r \\ a_i & r < i \leq n \end{cases}$$

becomes an orthonormal basis for the tangent space $T_p\Sigma$ and

$$e_{n+\alpha} = \begin{cases} \frac{1}{\sqrt{1+\lambda_\alpha^2}}(b_\alpha - \lambda_\alpha a_\alpha) & 1 \le \alpha \le r \\ b_\alpha & r < \alpha \le m \end{cases}$$

becomes an orthonormal basis for the normal space $N_p\Sigma$.

It is obvious that $r \le q := \min\{n, m\}$.

2.2 Weak Majorization

To control the singular values of the linear combination of two matrices, we will need the notion of weak majorization that is defined here.

Definition 2.1 Let $x, y \in \mathbb{R}^n$, we say that x is *weakly majorized* by y , denoted by $x \prec_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \le \sum_{i=1}^{k} y_i^{\downarrow} \qquad k = 1, 2, \dots, n$$

where $\{x_i^{\downarrow}\}$ is a rearrangement of $\{x_i\}$ in descending order, i.e. $x_1^{\downarrow} \ge \dots \ge x_n^{\downarrow}$.

For a fixed $y \in \mathbb{R}^n_{\ge 0}$, consider the set $W(y) := \{x \in \mathbb{R}^n_{\ge 0} : x \prec_w y\}$. And define $E(y)$ to be the set of vectors in $\mathbb{R}^n$ consisting of all $(\delta_1 y_{\pi(1)}, \dots, \delta_n y_{\pi(n)})$ where each δ_i takes the values of 0 or 1, and π is a permutation of $1, \dots, n$. So there are at most $2^n n!$ elements in $E(y)$. The relation between $W(y)$ and $E(y)$ is given by Mirsky [6] as follows:

Lemma 2.2 [6, Theorem 6] *The set $W(y)$ is the convex hull of the $E(y)$*

For $A, B \in M_{n\times m}$, there is a relation for the singular values λ of A, B and $A + B$ in term of majorization by Ky Fan [1].

Lemma 2.3 [1, Theorem 5] *For any $A, B \in M_{n\times m}$, we have $\lambda(A + B) \prec_w \lambda(A) + \lambda(B)$, where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ is the vector that consists of singular values of A. Because $\lambda_i = 0$ for $i > q = \min\{n, m\}$, we can rewrite $\lambda(A) = (\lambda_1, \dots, \lambda_q)$ instead.*

For completeness, we give a simple proof of Lemma 2.3.

Proof Let $A \in M_{n\times m}$, and consider its augmented matrix $\mathcal{A} = \begin{bmatrix} 0_n & A \\ A^\top & 0_m \end{bmatrix}$, where 0_n and 0_m represent the $n \times n$ zero matrix and $m \times m$ zero matrix, respectively. We denote the singular values of A as $0 \le \lambda_q \le \cdots \le \lambda_1$ where $q = \min\{n, m\}$. Since $\mathcal{A}$ is symmetric, $\mathcal{A}$ can be diagonalized. It can be shown that the eigenvalues of $\mathcal{A}$ are exactly $\lambda_i, -\lambda_i$ for $i = 1, 2.., q$ and additional $|m - n|$ zero eigenvalues. Assuming $n \ge m$, we in fact have the following relation by simple computation,

$$\det(\lambda I_{n+m} - \mathcal{A}) = \lambda^{n-m} \det\left(\lambda^2 I_n - AA^\top\right).$$

Thus, finding the sum of k-largest singular values of A is equivalent to finding the sum of k-largest eigenvalues of $\mathcal{A}$. The sum of k-largest eigenvalues of $\mathcal{A}$ is equal to

$$\sum_{i=1}^{k} \lambda_i = \max_{v_i \perp v_j, |v_i|=1} \sum_{i=1}^{k} v_i^{\top} \mathcal{A} v_i = \max_{V^{\top} V = I_k} \operatorname{tr}\left(V^{\top} \mathcal{A} V\right),$$

where $v_i \in M_{(n+m)\times 1}$ and $V \in M_{(n+m)\times k}$. Given $A, B \in M_{n\times m}$, we have the corresponding augmented matrices $\mathcal{A}, \mathcal{B} \in M_{n+m}$ and moreover,

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i(\mathcal{A}+\mathcal{B}) &= \max_{V^{\top} V = I_k} \operatorname{tr}\left(V^{\top}(\mathcal{A}+\mathcal{B})V\right) \\ &\leq \max_{V^{\top} V = I_k} \operatorname{tr}(V^{\top}\mathcal{A}V) + \max_{V^{\top} V = I_k} \operatorname{tr}(V^{\top}\mathcal{B}V) \\ &= \sum_{i=1}^{k} \lambda_i(\mathcal{A}) + \sum_{i=1}^{k} \lambda_i(\mathcal{B}). \end{aligned}$$

□

3 Proof of Main theorem

In this section, we will prove the main theorem

Theorem 3.1 *Suppose the graph of $f_0 : \Omega \subset M^n \to \mathbb{R}^m$ and $f_1 : \Omega \subset M^n \to \mathbb{R}^m$ are both minimal submanifolds in $M \times \mathbb{R}^m$ with $f_0|_{\partial\Omega} = f_1|_{\partial\Omega}$. If both the singular value vectors of f_0 and f_1 lie in a symmetric convex subset $\mathcal{C}$ of $\mathcal{M}$, then $f_0 = f_1$.*

Define $f_t(x) := tf_1(x) + (1-t)f_0(x)$ to be the geodesic homotopy joining f_0 and f_1. Fixing a point $p \in \Omega \subset M^n$, denote $df_1|_p$ and $df_0|_p$ by $m \times n$ matrices A and B, respectively. The condition of the theorem says that the singular value vectors of A, B satisfy $\lambda(A), \lambda(B) \in \mathcal{C}$, where $\lambda(A) := (\lambda_1(A), \ldots, \lambda_q(A))$ and $q = \min\{n, m\}$ is the singular value vector of A. Since $df_t = tdf_1 + (1-t)df_0$, to proceed the proof of the theorem we first need the following lemma

Lemma 3.2 *Let $A, B \in M_{m\times n}$ and their singular value vectors lie in a symmetric convex subset $\mathcal{C}$ of $\mathcal{M}$, i.e. $\lambda(A), \lambda(B) \in \mathcal{C}$. Then $\lambda(tA + (1-t)B) \in \mathcal{M}$ for any $t \in [0, 1]$.*

Proof Lemma 2.3 tells us that $\lambda(tA+(1-t)B) \prec_w t\lambda(A)+(1-t)\lambda(B)$, where we use the fact that $\lambda(tA) = t\lambda(A)$ for any $t \geq 0$. Lemma 2.2 gives a characterisation of weak majorization set that $\lambda(tA+(1-t)B)$ lies in the convex hull of $E(t\lambda(A)+(1-t)\lambda(B))$.

Claim If $\boldsymbol{x} = (x_1, x_2, \ldots, x_q) \in \mathcal{M}$, then $\boldsymbol{x}' = (x'_1, x'_2, \ldots, x'_q) \in \mathcal{M}$ for any $0 \leq x'_i \leq x_i$, $i = 1, 2, \ldots, q$.

Fix i and we only need to consider the case where $x'_k = x_k$ for all $k \neq i$. We can use induction for the general case. Let $x'_i = tx_i$ for some $t \in [0, 1]$. For $\boldsymbol{x} \in \mathcal{M}$, condition (1.5) is clearly preserved since if $x_i x_j < 1$ then $tx_i x_j < 1$ for any $t \in [0, 1]$. To check whether (1.6) is preserved, rewrite the expression (1.6) as

$$
\begin{aligned}
&\prod_{l=1}^{q}\left(1-x_l^2\right)+\sum_{l=1}^{q}\left(1-x_1^2\right)\cdots\left(1-x_{l-1}^2\right)x_l^2\left(1-x_{l+1}^2\right)\cdots\left(1-x_q^2\right)\\
&\quad=\left(1-x_1^2\right)\cdots\widehat{\left(1-x_i^2\right)}\cdots\left(1-x_q^2\right)\\
&\qquad+\sum_{l=1,l\neq i}^{q}\left(1-x_1^2\right)\cdots\left(1-x_{l-1}^2\right)x_l^2\left(1-x_{l+1}^2\right)\cdots\left(1-x_q^2\right)\\
&\quad=:P+\left(1-x_i^2\right)Q,
\end{aligned}
$$

where we denote $(1-x_i^2)Q=\sum_{l=1,l\neq i}^{q}(1-x_1^2)\cdots(1-x_{l-1}^2)x_l^2(1-x_{l+1}^2)\cdots(1-x_q^2)$ and $P=(1-x_1^2)\cdots\widehat{(1-x_i^2)}\cdots(1-x_q^2)$.

Since $\boldsymbol{x}\in\mathcal{M}$, the above expression is positive. If $P<0$, then $(1-x_i^2)Q>0$ and there exists $k\neq i$ such that $x_k>1$. Because $x_ix_k<1$, it implies $x_i<1$. Therefore $Q>0$ and $(1-t^2x_i^2)Q\geq(1-x_i^2)Q>0$ for $t\in[0,1]$. It follows that $P+(1-t^2x_i^2)Q>0$ and (1.6) holds. If $P\geq 0$, then the condition $x_kx_l<1$ for any $k\neq l$ gives $x_k\leq 1$ for all $k\neq i$ which implies $Q\geq 0$. Hence $P+(1-t^2x_i^2)Q\geq P+(1-x_i^2)Q>0$ for $t\in[0,1]$ and the claim is proved.

Next we show that the convex hull of $E(t\lambda(A)+(1-t)\lambda(B))$ lies in $\mathcal{M}$. Denote $z=t\lambda(A)+(1-t)\lambda(B)\in\mathcal{C}$. Since $\mathcal{C}$ is a symmetric convex subset in $\mathcal{M}$, the convex hull containing z and all permutation of coordinate of z must lie in $\mathcal{C}$ and hence in $\mathcal{M}$. The convex hull of $E(z)$, by definition, is exactly all the possible convex combination of any points of the form $(\delta_1z_{\pi(1)},\ldots,\delta_qz_{\pi(q)})$ where $\delta_i=0$ or 1, $z=(z_1,\ldots,z_q)$ and π is a permutation. We will show that the convex combination of any points of the form $(\delta_1z_{\pi(1)},\ldots,\delta_qz_{\pi(q)})$ must lie in $\mathcal{M}$. This will imply the convex hull is contained in $\mathcal{M}$. Let $p^1,\ldots,p^k$ be any k points of the form $(\delta_1z_{\pi(1)},\ldots,\delta_qz_{\pi(q)})$, and let $\bar{p}^1,\ldots,\bar{p}^k$ be the corresponding points with all $\delta_i=1$, so we have that $\bar{p}^1,\ldots,\bar{p}^k\in\mathcal{C}$. Consider any convex combination of $p^1,\ldots,p^k$, i.e. $\sum_{j=1}^{k}\alpha_jp^j$, where $\alpha_j\geq 0$ and $\sum_{j=1}^{k}\alpha_j=1$. By convexity and symmetry of $\mathcal{C}$, the point $\sum_{j=1}^{k}\alpha_j\bar{p}^j$ must lie in $\mathcal{C}$. Writing $\sum_{j=1}^{k}\alpha_jp^j=(a_1',\ldots,a_q')$ and $\sum_{j=1}^{k}\alpha_j\bar{p}^j=(a_1,\ldots,a_q)$, we can check that $0\leq a_i'\leq a_i$ for all i. Hence $\sum_{j=1}^{k}\alpha_jp^j\in\mathcal{M}$ by the claim, and the convex hull of $E(z)$ lies in $\mathcal{M}$. It thus concludes $\lambda(tA+(1-t)B)\in\mathcal{M}$ for any $t\in[0,1]$. □

We will show how uniqueness is obtained by analysing the second variational formula of area functional.

Proof of Main Theorem From the hypothesis, the singular value vectors of f_0 and f_1 lie in a symmetric convex subset $\mathcal{C}$ of $\mathcal{M}$, then Lemma 3.2 implies $\lambda(df_t)\in\mathcal{M}$ for $t\in[0,1]$. From [5, Theorem 4.1, Eq. (4.2)], we obtain the inequality for the second variational of area as

$$
\begin{aligned}
\frac{d^2 A}{dt^2} \geq \int_\Omega \sum_i \frac{1}{\left(1+\lambda_i^2\right)^2} \langle \nabla_{df(a_i)} V, b_i \rangle^2 \\
&+ \sum_{i \neq j} \frac{\lambda_i \lambda_j}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)} \langle \nabla_{df(a_i)} V, b_i \rangle \langle \nabla_{df(a_j)} V, b_j \rangle \\
&+ \int_\Omega \sum_{i \neq j} \frac{1}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)} \langle \nabla_{df(a_i)} V, b_j \rangle^2 \\
&- \sum_{i \neq j} \frac{\lambda_i \lambda_j}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)} \langle \nabla_{df(a_i)} V, b_j \rangle \langle \nabla_{df(a_j)} V, b_i \rangle,
\end{aligned}
$$

where $V = \frac{\partial f_t}{\partial t}$ is the variational vector field and a_i, b_i are defined as in Sect. 2 to be the orthonormal basis of pointwise singular values decomposition. Note that we have chosen f_t as geodesic homotopy so that $\nabla_V V = (\frac{d^2 f_t}{dt^2})^T = 0$ holds for any $t \in [0, 1]$.

Denote the above first two terms of the integrand as I, last two terms of the integrand by II and $p_{ij} := \langle \nabla_{df(a_i)} V, b_j \rangle$. First note that I > 0 if and only if $\mathcal{A} = \prod_{i=1}^n (1 + \lambda_i^2)^{\frac{1}{2}} = \prod_{i=1}^q (1 + \lambda_i^2)^{\frac{1}{2}}$ is strictly convex [3, Lemma 3.1]. Here we use that fact that $\lambda_i = 0$ for $i > q = \min\{n, m\}$ By collecting terms and factorising, II can be expressed as

$$
\begin{aligned}
\mathrm{II} = \frac{1}{2} \sum_{i \neq j} \frac{1}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)} \Big(&(|p_{ij}| - |p_{ji}|)^2 + 2(1 - \lambda_i \lambda_j) |p_{ij}||p_{ji}| \\
&+ 2\lambda_i \lambda_j (|p_{ij}||p_{ji}| - p_{ij} p_{ji}) \Big).
\end{aligned}
$$

Since $\lambda(df_t) \in \mathcal{M}$, this implies $\mathcal{A} = \prod_{i=1}^q (1 + \lambda_i^2)^{\frac{1}{2}}$ is strictly convex and both I, II ≥ 0. We can conclude that

$$
\frac{d^2 A}{dt^2} = \int_\Omega \mathrm{I} + \mathrm{II} \geq 0 \qquad \text{for any } t \in [0, 1]
$$

Moreover, the condition that f_0 and f_1 are minimal maps tells us $\frac{dA}{dt}|_{t=0} = 0 = \frac{dA}{dt}|_{t=1}$. Combining these two implies $\frac{d^2 A}{dt^2} = 0$ for all $t \in [0, 1]$. Hence I=II=0, forcing all $p_{ij} = 0$ for any i, j, and $\frac{df_t}{dt} \equiv V$ is a parallel vector field. As $f_0 = f_1$ on the boundary, we thus have $V \equiv 0$ on Ω and hence $f_0 = f_1$. □

4 Application

In this section, we discuss applications of our main theorem. First note that for the hypersurface case $q = \min\{n, m\} = 1$, and $\mathcal{M} = \mathbb{R}_{\geq 0}$. Thus we have

Corollary 4.1 *Assume $m = 1$ and the graphs of $f_0$andf_1 are both minimal submanifolds in $M \times \mathbb{R}$ with same boundary data. Then we have $f_0 \equiv f_1$.*

The next situation when $q = 2$, i.e. $n = 2$ or $m = 2$, the two- dimensional or codimension two minimal surfaces, we have $\mathcal{M} = \{\lambda_1\lambda_2 < 1\}$. The tangent line to the point $(1, 1)$ is $\lambda_1 + \lambda_2 = 2$. Hence the largest possible symmetric convex set in $\mathcal{M}$ can be described as $\lambda_1 + \lambda_2 < 2$, and we have

Corollary 4.2 *Assume* $\min\{n, m\} = 2$. *Suppose the graphs of* $f_0 and f_1$ *are both minimal submanifolds in* $M \times \mathbb{R}^m$ *with same boundary data and the singular values of* $f_0 and f_1$ *satisfy* $\lambda_1 + \lambda_2 < 2$, *then* $f_0 \equiv f_1$.

For $q = \min\{n, m\} > 2$, a natural attempt would be to consider the intersection of $\lambda_i + \lambda_j < 2$ for $i \neq j$ and (1.6). However, the intersection is not a convex set in $\mathcal{M}$ because of (1.6). We can consider the intersection of $\lambda_i + \lambda_j < 2$, $i \neq j$ and $\sum_{i=1}^{q} \lambda_i < 2 + \alpha$ instead, where $\alpha \simeq 0.568$ is the root of the cubic equation $2t^3 - 6t^2 + t + 1 = 0$. Then by Lagrangian Multiplier method, we can prove that it is a symmetric convex set in $\mathcal{M}$ and obtain the uniqueness result. In fact, the upper bound $2 + \alpha$ is the best constant in the sense that any larger upper bound will give a counterexample to (1.6). In the next corollary, we give a weaker but a more clean condition. The proof for the above claim is similar.

Corollary 4.3 *Suppose the graphs of* $f_0 and f_1$ *are both minimal submanifolds in* $M \times \mathbb{R}^m$ *with same boundary data and the singular values of* $f_0 and f_1$ *satisfy* $\lambda_1 + \cdots + \lambda_q < 2$, *where* $q = \min\{n, m\}$, *then* $f_0 \equiv f_1$.

Proof Let $S = \{\lambda : \lambda_1 + \cdots + \lambda_q < 2\}$. Clearly this set is symmetric and is convex. It remains to show that S lies in $\mathcal{M}$. Firstly, for any $\lambda \in S$, we have $2 > \lambda_i + \lambda_j \geq 2\sqrt{\lambda_i\lambda_j}$, so it satisfies (1.5). To check (1.6), we shall prove it via induction. Denote

$$F_q(\lambda_1, \ldots, \lambda_q) = \prod_{i=1}^{q}\left(1 - \lambda_i^2\right) + \sum_{i=1}^{q}\left(1 - \lambda_1^2\right)\cdots\left(1 - \lambda_{i-1}^2\right)\lambda_i^2\left(1 - \lambda_{i+1}^2\right)\cdots\left(1 - \lambda_q^2\right). \tag{4.1}$$

For $q = 2$, we have $F_2(\lambda_1, \lambda_2) = 1 - \lambda_1^2\lambda_2^2 > 0$. Assume that $F_{q-1} > 0$ is true. The condition $\lambda_i\lambda_j < 1$ for any $i \neq j$ tells us that at most one λ_i can be greater or equal to 1. If all $\lambda_i < 1$ then trivially $F_q > 0$. So WLOG we only need to consider the case $0 \leq \lambda_q \leq \cdots \leq \lambda_2 < 1 < \lambda_1$. If $\lambda_q = 0$ then by induction $F_q > 0$, so we can assume $\lambda_q > 0$. Condition (1.6) or equivalently $F_q > 0$ can be rewritten as

$$\sum_{k=2}^{q} \frac{\lambda_k^2}{1 - \lambda_k^2} < \frac{1}{\lambda_1^2 - 1}. \tag{4.2}$$

Since we have $(\lambda_1 + \lambda_q) + \lambda_2 + \cdots + \lambda_{q-1} < 2$, using induction on $\lambda_1 + \lambda_q, \lambda_2, \ldots, \lambda_{q-1}$, it follows that

$$\sum_{k=2}^{q-1} \frac{\lambda_k^2}{1-\lambda_k^2} < \frac{1}{(\lambda_1+\lambda_q)^2-1}.$$

Therefore by adding extra term on both sides, we get

$$\sum_{k=2}^{q} \frac{\lambda_k^2}{1-\lambda_k^2} < \frac{1}{(\lambda_1+\lambda_q)^2-1} + \frac{\lambda_q^2}{1-\lambda_q^2}. \tag{4.3}$$

We wish to show that RHS of Eq. (4.3) is less than $\frac{1}{\lambda_1^2-1}$ or equivalently

$$\frac{1}{\left(\lambda_1+\lambda_q\right)^2-1} < \frac{1-\lambda_1^2\lambda_q^2}{\left(1-\lambda_q^2\right)\left(\lambda_1^2-1\right)} \iff \lambda_1^2\lambda_q^2\left(\lambda_1+\lambda_q\right)^2 < 2\lambda_1\lambda_q\left(\lambda_1\lambda_q+1\right).$$

But the last inequality follows from $\lambda_1, \lambda_q > 0$, $\lambda_1 + \lambda_q < 2$ and $\lambda_1\lambda_q < 1$ because

$$\lambda_1\lambda_q\left(\lambda_1+\lambda_q\right)^2 < 4\lambda_1\lambda_q < 2\left(\lambda_1\lambda_q+1\right).$$

Hence we have shown that $S \subset \mathcal{M}$ and our main theorem can be applied. □

Next we continue to give various different symmetric convex sets in $\mathcal{M}$ and obtain uniqueness results in these situations. None of these sets is a proper subset of another as discussed in Remark 4.7.

Observe that the set of distance decreasing map, i.e. each singular value is less than one, is a symmetric convex set in $\mathcal{M}$. Hence we have the following.

Corollary 4.4 *Suppose the graphs of $f_0$andf_1 are both minimal submanifolds in $M \times \mathbb{R}^m$ with same boundary data and the maps are distance decreasing, then $f_0 \equiv f_1$.*

The next example comes from [3]. From the proof of Theorem 4.6 in [3], they have basically shown that the area functional $\sqrt{\prod_{i=1}^q (1+\lambda_i^2)}$ is strictly convex in the sublevel set $\sqrt{\prod_{i=1}^q (1+\lambda_i^2)} < 2$. This implies that the sublevel set $\prod_{i=1}^q (1+\lambda_i^2) < 4$, is convex and is a subset of $\mathcal{M}$. Thus we obtain the following uniqueness result.

Corollary 4.5 *Suppose the graphs of $f_0$andf_1 are both minimal submanifolds in $M \times \mathbb{R}^m$ with same boundary data and the singular values of f_0 and f_1 satisfy $\prod_{i=1}^q (1+\lambda_i^2) < 4$, where $q = \min\{n, m\}$, then $f_0 \equiv f_1$.*

Lastly, we show that a ball of radius $\sqrt{2}$ is the largest possible ball centred at origin and lying in $\mathcal{M}$. It can be deduced easily from the condition $\lambda_i\lambda_j < 1$ that $\sqrt{2}$ is the best constant.

Corollary 4.6 *Suppose the graphs of* $f_0 and f_1$ *are both minimal submanifolds in* $M \times \mathbb{R}^m$ *with same boundary data and their singular values satisfy* $\sum_{i=1}^{q} \lambda_i^2 < 2$, *where* $q = \min\{n, m\}$, *then* $f_0 \equiv f_1$.

Proof We use similar induction idea as in the proof of Corollary 4.3. First note that the set $\left\{\lambda : \sum_{i=1}^{q} \lambda_i^2 < 2\right\}$ satisfies $\lambda_i \lambda_j \le \frac{\lambda_i^2+\lambda_j^2}{2} < 1$ for any $i \neq j$. This set is clearly symmetric and convex. Assume that $F_{q-1} > 0$ is true. We only need to consider the case $0 < \lambda_q \le \cdots \le \lambda_2 < 1 < \lambda_1$. Using induction on $\sqrt{\lambda_1^2 + \lambda_q^2}, \lambda_2, \ldots, \lambda_{q-1}$. which satisfies $\sum_{i=1}^{q} \lambda_i^2 < 2$, then by induction hypothesis,

$$\sum_{k=2}^{q-1} \frac{\lambda_k^2}{1-\lambda_k^2} < \frac{1}{\left(\sqrt{\lambda_1^2+\lambda_q^2}\right)^2 - 1}$$

adding extra term on both sides we get

$$\sum_{k=2}^{q} \frac{\lambda_k^2}{1-\lambda_k^2} < \frac{1}{\lambda_1^2+\lambda_q^2-1} + \frac{\lambda_q^2}{1-\lambda_q^2}. \tag{4.4}$$

We wish to show that RHS of Eq. (4.4) is less than $\frac{1}{\lambda_1^2-1}$ or equivalently

$$\frac{1}{\lambda_1^2+\lambda_q^2-1} < \frac{1-\lambda_1^2\lambda_q^2}{\left(1-\lambda_q^2\right)\left(\lambda_1^2-1\right)} \iff \lambda_1^2\lambda_q^2\left(\lambda_1^2+\lambda_q^2\right) < 2\lambda_1^2\lambda_q^2$$

which is clearly true. □

Remark 4.7 Denote

$$\begin{aligned}
A &= \{\lambda | \lambda_i < 1, i = 1, \ldots, q\} \\
B &= \left\{\lambda | \sum_{i=1}^{q} \lambda_i < 2\right\} \\
C &= \left\{\lambda | \prod_{i=1}^{q} (1+\lambda_i^2) < 4\right\} \\
D &= \left\{\lambda | \sum_{i=1}^{q} \lambda_i^2 < 2\right\}.
\end{aligned}$$

We take $q = 3$ as an example to show that the above sets do not contain each other. Let $\epsilon > 0$ be a small number, then we can check the following

(1) $x = (1-\epsilon, 1-\epsilon, 1-\epsilon) \in A$, but $\notin B, C, D$.
(2) $x = (2-\epsilon, \epsilon/3, \epsilon/3) \in B$, but $\notin A, C, D$.

(3) $x = (\sqrt{3-\epsilon}, \sqrt{\epsilon}/3, \sqrt{\epsilon}/3) \in C$, but $\notin A, D$ while
$x = (\sqrt{\sqrt[3]{4}-1-\epsilon}, \sqrt{\sqrt[3]{4}-1-\epsilon}, \sqrt{\sqrt[3]{4}-1-\epsilon}) \in C$, but $\notin B$.
(4) $x = (\sqrt{1-2\epsilon}, \sqrt{1-2\epsilon}, \sqrt{3\epsilon}) \in D$, but $\notin B, C$ whereas
$x = (\sqrt{2-\epsilon}, \sqrt{\epsilon}/2, \sqrt{\epsilon}/2) \in D$, but $\notin A$.

5 Uniqueness Results in General Riemannian Manifold

In this section, we discuss a generalisation of the uniqueness result when N has non-positive sectional curvature.

Suppose f_0 and f_1 are homotopic. Then we can lift the homotopy map of $f_0 and f_1$ to the universal covering of N. By Hadamard theorem, the universal covering of N is diffeomorphic to $\mathbb{R}^m$ and hence there exists a unique geodesic connecting the lifting $\tilde{f}_0(x)$ and $\tilde{f}_1(x)$. Denote the projection of this unique geodesic onto N by $\gamma_x(t)$ and define $f_t(x) := \gamma_x(t)$.

Fix a point $p \in M$ with an orthonormal basis $\{\frac{\partial}{\partial x^i}\}$, then $J_{i,p}(t) := Df_t(\frac{\partial}{\partial x^i})$ is a Jacobi field. It can be checked that $\|J_{i,p}(t)\|^2$ is a convex function of t (refer to [5, Equation (3.3)]) and hence so is $\sum_{i=1}^n \|J_{i,p}(t)\|^2$. The convexity of this function is essential in the proof of uniqueness of minimal graph in [5] in order to control the singular values of the homotopy map f_t.

On the other hand, we prove in the following lemma the relation $\sum_{i=1}^n \|J_{i,p}(t)\|^2 = \sum_{i=1}^n \lambda_i^2$ where λ_i are the singular values of $Df_t(p)$.

Lemma 5.1 $\sum\limits_{i=1}^n \|J_{i,p}(t)\|^2 = \sum\limits_{i=1}^n \lambda_i^2$

Proof

$$\begin{aligned}
\sum_{i=1}^n \|J_{i,p}(t)\|^2 &= \sum_{i=1}^n \left\langle Df_t\left(\frac{\partial}{\partial x^i}\right), Df_t\left(\frac{\partial}{\partial x^i}\right)\right\rangle \\
&= \sum_{i=1}^n \left\langle (Df_t)^\top Df_t\left(\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^i}\right\rangle \\
&= \sum_{i=1}^n \left\langle (Df_t)^\top Df_t\left(\sum_j c_i^j a_j\right), \sum_j c_i^j a_j\right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i^j \lambda_j^2 c_i^j \\
&= \sum_{i=1}^n \lambda_i^2
\end{aligned}$$

where $\{a_i\}$ is the singular value basis for $Df_t(p)$ corresponding to $\{\lambda_i\}$ and $\sum_j (c_i^j)^2 = 1$ □

Using this lemma, we are able to generalise Corollary 4.6 as follows.

Theorem 5.2 *Assume N has non-positive sectional curvature. Suppose the graphs of f_0 and f_1 are both minimal submanifolds in $M \times N$ with the same boundary data. If f_0 and f_1 are homotopic and their singular values satisfy $\sum_{i=1}^{q} \lambda_i^2 < 2$, then $f_0 \equiv f_1$.*

Proof From the lemma, we know that $\sum_{i=1}^{n} \lambda_i^2$ can be represented as sum of square of Jacobi fields which is a convex function of t. Since both f_0 and f_1 satisfy $\sum_{i=1}^{q} \lambda_i^2 < 2$, convexity tells us that the singular values of f_t for any $t \in (0, 1)$ also satisfy $\sum_{i=1}^{q} \lambda_i^2 < 2$, i.e. $\lambda(df_t) \in \mathcal{M}$ for any $t \in [0, 1]$. Following the same proof as in our main theorem, we obtain $V = 0$ where $V = \frac{df_t}{dt}$ and hence the uniqueness result is achieved. □

Acknowledgements The first two authors are supported by 102-2115-M-002-013-MY3, and the last author is supported by 105-2115-M-002-006-MY2. The second author would like to thank Kai-Wei Zhao and Wei-Bo Su for their interests and discussions.

References

1. Fan, K.: Maximum properties and inequalities for the eigenvalues of completely continuous operators. Proc. Natl. Acad. Sci. USA **37**, 760–766 (1951)
2. Lawson Jr., H.B., Osserman, R.: Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system. Acta Math. **139**(1–2), 1–17 (1977)
3. Lee, Y.-I., Tsui, M.-P.: Stability of the minimal surface system and convexity of area functional. Trans. Am. Math. Soc. **366**(7), 3357–3371 (2014)
4. Lee, Y.-I., Wang, M.-T.: A stability criterion for nonparametric minimal submanifolds. Manuscr. Math. **112**(2), 161–169 (2003)
5. Lee, Y.-I., Wang, M.-T.: A note on the stability and uniqueness for solutions to the minimal surface system. Math. Res. Lett. **15**(1), 197–206 (2008)
6. Mirsky, L.: On a convex set of matrices. Arch. Math. **10**, 88–92 (1959)