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Higher Codimension Minimal Submanifold with Isolated Singularity

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Abstract

We study the existence of higher codimension minimal submanifold with isolated singularity but is not a cone. We generalize the fix point method in Caffarelli et al. (Manuscripta Math 48(1–3):1–18, 1984) to higher codimension setting and show the existence of non-conical higher codimension minimal submanifold (with boundary) with isolated singularity.

Keywords Minimal submanifold · Higher codimension · Isolated-singularity

Mathematics Subject Classification 53A07

1 Introduction

The study of singularity of higher codimension minimal submanifold is an important research field in differential geometry. In this article, we shall focus on minimal submanifold with isolated singularity. One important example is Lawson-Osserman minimal cone constructed in [7]. Lawson-Osserman constructed it using Hopf map $\eta : \mathbb{S}^3 \subset \mathbb{C}^2 \to \mathbb{S}^2 \subset \mathbb{R} \times \mathbb{C}$ which is given by

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z_2})$$

and the corresponding minimal cone is

$$\{(2x/3, \sqrt{5}\eta(x)/3) \in \mathbb{R}^4 \times \mathbb{R}^3 : \|x\| = 1\} \subset \mathbb{S}^6 \tag{1}$$

A calibration argument by Harvey–Lawson says that this minimal cone is a coassociative submanifold which implies that it is area-minimizing [5]. Recently, by generalizing

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the anszats in (1) to more general harmonic map instead of Hopf map, Xu-Yang-Zhang are able to construct countably many distinct higher codimension minimal cone [15] . Furthermore in some cases, they are able to perturb the cone to produce smooth entire minimal submanifold with the minimal cone as the tangent cone at infinity.

A general way to obtain minimal submanifold with isolated singularity in Euclidean space is to consider the cone over a minimal submanifold in sphere. Let M^{n-1} be a smooth embedded minimal submanifold of $\mathbb{S}^{n+k-1}$ which is non-totally geodesic. We define its cone to be

$$C = C(M) := \{tx : t > 0, x \in M\}$$

which is singular only at the origin. It is well known that C is minimal in $\mathbb{R}^{n+k}$ if and only if M is minimal in $\mathbb{S}^{n+k-1}$. A classical example of this construction is the Simon cone $C_{p,q}$ which is a cone over minimal hypersurface $M_{p,q}$ in $\mathbb{S}^{p+q+1}$ [8], where we define for $p, q \in \mathbb{Z}^+$,

$$M_{p,q} := \mathbb{S}^p\left(\sqrt{\frac{p}{p+q}}\right) \times \mathbb{S}^q\left(\sqrt{\frac{q}{p+q}}\right)$$

and

$$C_{p,q} := C(M_{p,q})$$

Minimal cone is a very important geometric model in analyzing the singularity of a minimal submanifold. In the context of Geometric measure theory, the regularity of codimension one area-minimizing current tells us that the Hausdorff dimension of the singular set of an area-minimizing n-current is at most $n-7$, where n is dimension of the minimizing current (see for example Chapter 7 of [9]). An immediate consequence of this regularity theory is any area-minimizing hyper-current of dimension 6 or less must be smooth. The Simon cone $C_{3,3}$ defined above for $p = q = 3$ is known to be an example of area-minimizing current with isolated singularity in $\mathbb{R}^8$ by Bombieri-Giorgi-Giusti [2]. One can now ask the question if the singular set of a minimizing current has Hausdorff dimension 0, or equivalently when the singularities are isolated, what are the possible example other than the cone. In [3], Caffarelli-Hardt-Simon answered this by constructing minimal hypersurface (with boundary) with isolated singularities but which is not a minimal cone. They study an operator called the mean curvature operator defined on a truncated minimal cone and use perturbation of eigenfuntions of its linearized operator to produce singular solution for a non-linear PDE. If furthermore we know that the cone is strictly minimizing (a condition defined in [6] which is stronger than minimizing current), then the perturbation is minimizing as well [6]. Later on, Smale generalizes Caffarelli-Hardt-Simon's construction and produces new examples of minimal hypersurfaces with more than one isolated singularities [11–13].

In higher codimension, the regularity theory of Almgren tells us that the singular sets of area minimizing current has Hausdorff dimension at most $n-2$ [1]. We can still ask the same question whether there is any example of minimal submanifold with isolated singularities but is not a cone. Indeed we show that Caffarelli-Hardt-Simon result can be generalized to higher codimension, that is we can construct higher codimension minimal submanifold with isolated singularity but is not a minimal cone. At the end of their paper, although Caffarelli-Hardt-Simon [3] did mention that their method can be generalized to higher codimension but no clear argument is provided. We shall attempt to write down a proof in the higher codimension case.

We first introduce some notation and background for this problem. Let C be n-dimension cone in $\mathbb{R}^{n+k}$ with vertex at the origin, this means

$$\lambda C = C, \quad \text{for all } \lambda > 0$$

Consider $G_\Phi(C)$ a graph over C w.r.t a map $\Phi : C \to NC$ i.e.

$$G_\Phi(C) := \{x + \Phi(x) : x \in C\} \subset \mathbb{R}^{n+k}$$

where NC is normal bundle of C. Such $G_\Phi(C)$ is also called a perturbation of the cone C in $\mathbb{R}^{n+k}$.

Let $\bar{\nabla} : \Gamma(TC) \times \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and $\nabla^\perp : \Gamma(TC) \times \Gamma(NC) \to \Gamma(NC)$ be the Euclidean connection and the normal connection on C respectively. We also denote $H_G(\Phi)$ as the mean curvature vector of $G_\Phi(C)$. The question that we are interested in is what Φ can give us $H_G(\Phi) = 0$ while also satisfying $G_\Phi(C)$ not a cone. Recall that $H_G(\Phi)$ is a section of normal bundle of $G_\Phi(C)$, so we cannot view $H_G(.)$ as an operator from on $\Gamma(NC)$ to $\Gamma(NC)$. Instead, we consider the projection of $H_G(\Phi)$ onto the NC which we denote it as the mean curvature operator $M_G(\Phi)$,

$$M_G(\Phi) := H_G(\Phi)^{\perp_{NC}}$$

If furthermore we know that C is a minimal and C^1 norm of Φ is small i.e.

$$|\Phi/r| < 1, \quad |\nabla^\perp \Phi| < 1,$$

then by Taylor expansion, $M_G(\Phi)$ can be written as (see [3, 11]) ,

$$M_G(\Phi) = \Delta^\perp \Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij} + Q(\Phi/r, \nabla^\perp \Phi, (\nabla^\perp)^2 \Phi) \tag{2}$$

where $A_{ij} = \nabla^\perp_{e_i} e_j$ is second fundamental form on C w.r.t some orthonormal basis $\{e_i\}$, $\Delta^\perp = \sum_i \nabla^\perp_{e_i} \nabla^\perp_{e_i} - \nabla^\perp_{(\nabla_{e_i} e_i)^T}$ is the normal laplacian, $r = |x|$ with $x \in C \subset \mathbb{R}^{n+k}$ and Q is higher order term. For any unit vector $V \in NC$, we can view the higher order term $Q^V := \langle Q, V \rangle$ as a polynomial $Q^V(w/r, z, p)$ with $w \in \mathbb{R}^n$, $z \in \mathbb{R}^{nN}$ and $p \in \mathbb{R}^{n^2 N}$ where $N = n + k$. Q^V has the properties that it is at least quadratic, and is linear in p. Moreover it satisfies

$$|Q^V| \leq \mu \left(\frac{1}{r} (|w/r|^2 + |z|^2) + (|w/r| + |z|)|p| \right) \tag{3}$$

$$|D_w Q^V|, |D_z Q^V| \leq \mu \left(\frac{1}{r}(|w/r| + |z|) + |p| \right) \tag{4}$$

$$|D_p Q^V| \leq \mu\,(|w/r| + |z|) \tag{5}$$

for some constant μ.

Now we shall state what problem we want to solve:
Problem Given a minimal cone C in a Euclidean space, can we find a small perturbation of the cone to produce a minimal submanifold with the same isolated singularity as the cone but is not itself a cone? This is same as asking whether we can find Φ in some suitable function space such that $M_G(\Phi) = 0$.

Remark It is known in [10,page 515], that the condition $M_G(\Phi) = 0$ is sufficient to imply $H_G(\Phi)^{\perp_{NC}} = 0$. Roughly speaking, if $|\nabla\Phi|$ is small enough, then the two fiber $N_x C$ and $N_x G_\Phi(C)$ are close to each other.

To tackle this problem, we first study the linearized part of the operator M_G in which we denote as L, an elliptic operator acting on section of NC which and is defined as

$$L\Phi = \Delta^\perp \Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij}$$

where A_{ij} is the second fundamental form of C w.r.t some orthornomal bases $\{e_i\}$. L is also known as the Jacobi operator in the literature. If C is a cone over a smooth manifold M, i.e.

$$C := \{rx : r > 0, x \in M \subset \mathbb{R}^{n+k}\}$$

we can decompose the operator $\Delta^\perp$ using geodesic polar coordinate

$$\Delta^\perp \Phi = r^{1-n} \frac{\partial}{\partial r} (r^{n-1} \frac{\partial \Phi}{\partial r}) + \frac{1}{r^2} \Delta^\perp_M \Phi$$

We also have the relation of second fundamental form A_{ij} on $C \subset \mathbb{R}^{n+k}$ and second fundamental form A^M_{ij} on $M \subset S^{n+k-1}$ which is $A_{ij}(rx) = \frac{1}{r} A^M_{ij}(x)$ where $x \in M, r > 0$ (refer to Section 1.4 of [14]) . Then L can be written as

$$L\Phi = r^{1-n} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} T\Phi$$

where $T\Phi = \Delta^\perp_M \Phi + \langle \Phi, A^M_{ij} \rangle A^M_{ij}$.

Our goal is to find Φ such that $M_G(\Phi) = 0$ and Φ satisfies certain decay at origin. The strategy is to first solve the Dirichlet problem of the linearized equation $L\Phi = F$ on $C^*_1 := C \cap B_1(0) \backslash \{0\}$ with boundary condition $\Phi = H$ on M for any suitably data $F \in L^2(C), H \in L^2(M)$. This can be done by considering eigenfunctions expansion of L^2 orthonormal basis of $L^2(M; NC)$ and hence reducing the problem into an ODE problem.

After that we introduce suitable weighted Holder space and try to prove regularity of the L^2 solution obtained in that space if F has better regularity. The final step is using fix point method to solve the nonlinear problem.

We shall now state the main result of this article which basically says that we can perturb a higher codimension minimal cone in Euclidean space to produce a minimal submanifold with the same isolated singularity as the minimal cone. Moreover, we can also obtain its asymptotic behaviour near the singularity. The statement of the result is as follow:

Theorem 1 *(Theorem 12) Let M^{n-1} be a smooth minimal submanifold in $\mathbb{S}^{n+k-1}$ and C be cone over M. We also denote $C_1^* = C\backslash\{0\} \cap B_1(0)$ as the truncated cone with a unit ball in $\mathbb{R}^{n+k}$. Given $\alpha \in (0,1)$, $\nu > 3$ and $J \geq 1$ such that $\Re\gamma_J < \nu < \Re\gamma_{J+1}$, there exist positive constants A and ϵ_0 (depending only on M, n, k, ν, α) such that corresponding to any $H \in C^{2,\alpha}(M)$ with $||\Pi_J H||_{C^{2,\alpha}(M)} < \epsilon_0$, there exist $\Phi \in C_\nu^{2,\alpha}(C_1^*)$, a $C^{2,\alpha}$ section of the normal bundle of C_1^* in $\mathbb{R}^{n+k}$ with decay rate $O(|x|^\nu)$ near origin, satisfying the zero mean curvature operator equation (2) i.e.*

$$M_G(\Phi) = 0 \text{ on } C_1^*$$

with boundary data H in projection sense,

$$\Pi_J \Phi = \Pi_J H \quad \text{on } M.$$

Moreover this solution satisfies the estimate

$$||\Phi||_{C_\nu^{2,\alpha}(C_1^*)} \leq A||\Pi_J H||_{C^{2,\alpha}(M)}$$

Remark The precise definition of J, γ_J and Π_J can be found in Sect. 2, whereas the weighted Holder space $C_\nu^{2,\alpha}$ is defined in Sect. 3.

The proof basically follows the codimension one case but we need to take care of the issue of a PDE system. Most of the C^k and Holder estimate in [3] is only being stated without giving detail proof, so generalizing those estimate to elliptic system is not an easy task. The approach I take is first reduce the elliptic system to some scalar equation, and then apply scalar Schauder theory together with appropriate interpolation inequality.

The organization of this paper is as follow, in Sect. 2, we construct the L^2 solution of the linearization of mean curvature operator using eigenfunction decomposition. This part generalizes directly from codimension one. In Sect. 3, we define the appropriate weighted Holder space for our setting and try to estimate the weighted Holder norm of the solution of the linearized equation. The main difficulty is how to do rescaling estimate on the minimal cone in Euclidean space. Finally, we shall apply fix point theorem to solve the nonlinear mean curvature equation in Sect. 4.

2 L^2 Solution of Linearized Equation

Let C^n be n-dimensional cone over $M^{n-1} \subset \mathbb{S}^{n+k-1}$ which is also a submanifold of $\mathbb{R}^{n+k}$. As a Riemannian manifold, the truncated cone $C\backslash\{0\}$ can also be viewed as the embedding of the product manifold $(0, \infty) \times M$ into Euclidean space $\mathbb{R}^{n+k}$ given by

$$\begin{aligned} \iota : (0, \infty) \times M &\to (\mathbb{R}^{n+k}, g_0) \\ (t, x) &\mapsto tx \end{aligned}$$

with induced metric $g_c := \iota^* g_0 = dt^2 + t^2 g_M$, where g_M is the Riemannian metric on M and g_0 is the Euclidean metric on $\mathbb{R}^{n+k}$. So $C\backslash\{0\} = \iota((0, \infty) \times M)$ and we shall call such M the link of the cone C. We also define the truncated cone C_r^* as the intersection of $C\backslash\{0\}$ with ball of radius r $B_r(0) \subset \mathbb{R}^{n+k}$ i.e.

$$C_r^* := C\backslash\{0\} \cap B_r(0)$$

In particular for $r = 1$, M will be the boundary of C_1^*.

Next we define the space $L^2(C_1^*; NC)$, where NC is the normal bundle of C, to be the space consisting of L^2 vector-valued function $f \in L^2(C_1^*; \mathbb{R}^{n+k})$ such that

$$f(x) \in N_x C \quad \forall x \in C_1^*$$

Given $F \in L^2(C_1^*; NC)$ and $H \in L^2(M; NC)$, we consider the Dirichlet problem on truncated cone

$$\begin{aligned} L\Phi &= F \text{ on } C_1^* \\ \Phi &= H \text{ on } M \end{aligned}$$

where the linear operator L acting on $L^2(((0, 1] \times M, g_c), NC)$ is defined as:

$$L\Phi = \left(r^{1-n} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial \Phi}{\partial r} \right) \right)^{\perp} + \frac{1}{r^2} T\Phi$$

and $T\Phi := \Delta_M^{\perp} \Phi + \langle \Phi, A_{ij}^M \rangle A_{ij}^M$. T is a compact self-adjoint operator acting on $L^2(M; NC)$, by spectral decomposition of $L^2(M; NC)$, there exist L^2 orthonormal eigenfunction ϕ_i and its corresponding eigenvalue μ_i

$$\begin{aligned} &\mu_1 \le \mu_2 \le \cdots \to \infty \\ &\phi_1, \phi_2, \cdots \in L^2(M; NC) \end{aligned}$$

satisfying $T\phi_i = -\mu_i \phi_i$. Using this L^2 basis $\{\phi_j\}_j$, we can expand $\Phi(r, x) = \sum a_i(r)\phi_i(x)$ and $F(r, x) = \sum f_i(r)\phi_i(x)$ for $x \in M$, $0 < r \le 1$ where

$$a_i(r) \equiv \langle \Phi, \phi_i \rangle_{L^2} := \int_M \langle \Phi(r, x), \phi_i(x) \rangle dV_x$$

and

$$f_i(r) \equiv \langle F, \phi_i \rangle_{L^2} := \int_M \langle F(r,x), \phi_i(x) \rangle dV_x$$

The inner product in the integral is the usual Euclidean inner product. By taking L^2 inner product with ϕ_j both side,

$$\langle L\Phi, \phi_j \rangle_{L^2} = \langle F, \phi_j \rangle_{L^2}$$

the linearized equation can be simplified into an ODE:

$$r^2 a_j''(r) + (n-1) r a_j'(r) - \mu_j a_j(r) = r^2 f_j(r) \tag{6}$$

We first solve the homogeneous equation

$$r^2 a_j''(r) + (n-1) r a_j'(r) - \mu_j a_j(r) = 0 \tag{7}$$

If we let $a_j(r) = c r^{\gamma_j}$ for any constant c, and plug in into (7), then we obtain

$$r^{\gamma_j} (\gamma_j^2 + (n-2)\gamma_j - \mu_j) = 0$$

If we choose γ_j to satisfy the quadratic equation above, then $a_j(r) = c r^{\gamma_j}$ is a solution to (7). In particular choose the root with plus sign

$$\gamma_j = -\frac{1}{2}(n-2) + \sqrt{\frac{(n-2)^2}{4} + \mu_j} \tag{8}$$

we remark that γ_j could be complex valued since μ_1 might not be non-negative.

Remark In codimension one, the condition $\frac{(n-2)^2}{4} + \mu_1 \geq 0$ corresponds to the cone C being stable [3]. However, it is not clear what is the analogue condition in higher codimension.

In order to obtain particular solution, we use variation of parameter by letting $a_j(r) = c_j(r) r^{\gamma_j}$ and then plugging back into (6). We then obtain

$$c_j'' r^{\gamma_j} + c_j'(2\gamma_j + (n-1)) r^{\gamma_j - 1} = f_j$$

this simplifies to

$$(r^{n-1+2\gamma_j} c_j')' = r^{n-1+\gamma_j} f_j$$

Both the real part and imaginary part of such a_j will satisfy (6). We chose the real part of a_j and the formal general solution for (6) is

$$a_j(r) = \Re\left(\alpha_j r^{\gamma_j} + r^{\gamma_j} \int_{\beta_j}^{r} s^{1-n-2\gamma_j} \int_0^s t^{n-1+\gamma_j} f_j(t)dtds\right)$$

for some constant α_j, β_j. Next, fix $\nu > 1$ and we would like to prescribe a decay rate r^ν for the solution near the origin. The reason for $\nu > 1$ is so that it decays faster than cone solution which correspond to decay rate $O(r)$. We choose $J \geq 1$ to satisfy the following

$$\Re\gamma_J < \nu < \Re\gamma_{J+1} \tag{9}$$

In order to make sense of the formal solution, some conditions are needed. By definition, $\Re\gamma_j \geq -(n-2)/2$ for any j. If we assume $|f_j| \leq cr^{\nu-2}$, then the integral in t makes sense since $(n-1+\Re\gamma_j+\nu-2) > -1$. Next, since $n-1+2\Re\gamma_j \geq 1$, we have $r^{n-1+2\gamma_j}c_j' \to 0$ when $r \to 0^+$, so the lower limit of the integral in t matches the formula. As for the integral in s, we are dealing with the function $s^{\nu-1-\gamma_j}$ which is also integrable from our choice of ν and γ_J. In order to have $a_j = O(r^\nu)$, we define it as follow:

$$a_j(r) = \begin{cases} \Re\left(r^{\gamma_j} \int_0^r s^{1-n-2\gamma_j} \int_0^s t^{n-1+\gamma_j} f_j(t)dtds\right) & j \leq J \\ \Re\left(\alpha_j r^{\gamma_j} + r^{\gamma_j} \int_1^r s^{1-n-2\gamma_j} \int_0^s t^{n-1+\gamma_j} f_j(t)dtds\right) & j \geq J+1 \end{cases} \tag{10}$$

where we have chosen $\alpha_j = \beta_j = 0$ for $j \leq J$ and $\alpha_j \in \mathbb{R}$, with $\sum \alpha_j^2 < \infty$, $\beta_j = 1$ for $j \geq J+1$. By direct computation, we can then check that

$$|r^{\gamma_j} \int_{\beta_j}^r s^{1-n-2\gamma_j} \int_0^s t^{n-1+\gamma_j} f_j(t)dtds| \leq cr^\nu \left|1 - \left(\frac{\beta_j}{r}\right)^{\nu-\Re\gamma_j}\right|$$

which implies $a_j(r) = O(r^\nu)$ when $r \to 0^+$.

On the other hand, the boundary behavior of Φ can be seen by letting $r \to 1$, then

$$\begin{aligned} H(x) &= \sum a_j(1)\phi_j(x) \\ &= \sum_{j=1}^{J} \Re\left(\int_0^1 s^{1-n-2\gamma_j} \int_0^s t^{n-1+\gamma_j} f_j(t)dtds\right)\phi_j(x) \\ &\quad + \sum_{j=J+1}^{\infty} \alpha_j\phi_j(x) \end{aligned}$$

this tells us that the first J term of the L^2 solution is determined by F, so we can only prescribed the term from $J+1$ onwards for H i.e.

$$\alpha_j = \langle H, \phi_j \rangle_{L^2}, \quad j \geq J+1$$

Define the projection operator

$$\Pi_J : L^2(M) \to L^2(M), \quad \Pi_J H := \sum_{j=J+1}^{\infty} \langle H, \phi_j \rangle_{L^2} \phi_j$$

The above discussion says that we can solve the following Dirichlet problem in L^2

$$\begin{aligned} L\Phi &= F \text{ on } C_1^* \\ \Pi_j \Phi &= \Pi_J H \text{ on } M \end{aligned}$$

where $F \in L^2(C_1)$ satisfies $||r^{2-\nu} F||_{L^\infty(C_1)} \leq c$, $H \in L^2(M)$. Moreover, we can estimate the L^2 norm of $\Phi = \sum_j a_j \phi_j$ using Parserval identity:

$$\begin{aligned} ||\Phi(r,.)||^2_{L^2(M)} &= \sum_{j \leq J} |a_j(r)|^2 + \sum_{j>J} |a_j(r)|^2 \\ &\leq Cr^{2\nu} ||r^{2-\nu} F||^2_\infty + \sum_{j \geq J+1} \alpha_j^2 |r^{2\gamma_j}| \\ &\leq Cr^{2\nu} ||r^{2-\nu} F||^2_\infty + r^{2\nu} \sum_{j \geq J+1} \alpha_j^2 \end{aligned}$$

Hence we have

$$||\Phi(r,.)||_{L^2(M)} \leq Cr^\nu \left(||r^{2-\nu} F||_\infty + \sqrt{\sum_{j>J} \alpha_j^2} \right) \tag{11}$$

This means the solution Φ decays like r^ν near origin in L^2 norm. With a given boundary data and decay rate, this solution can be proven to be unique (see for example [11,Lemma 3.3]).

We summarize the result we obtain as follows:

Theorem 2 *Let $\nu > 1$ and J satisfies (9). For any $H \in L^2(M)$ and F satisfying $||r^{2-\nu} F||_{L^\infty(C_1)} < \infty$, there exist constant $C > 0$ and unique solution $\Phi \in L^2(C_1)$ to the equation*

$$\begin{aligned} L\Phi &= F \textit{ on } C_1^* \\ \Pi_j \Phi &= \Pi_J H \textit{ on } M \end{aligned}$$

Moreover, for $0 < r < 1$, Φ satisfies the inequality

$$||\Phi(r,.)||_{L^2(M)} \leq Cr^\nu \left(||r^{2-\nu} F||_\infty + ||\Pi_J H||_{L^2(M)} \right)$$

3 Weighted Holder Estimate

After solving the linearized equation in L^2, we would like to use fix point method to solve the nonlinear equation. In order to do so, we need a better Banach space than L^2. A suitable setting would be in some weighted Holder space as we will define below.

3.1 Weighted Holder Space on C_1^*

Let $(\mathcal{U}_l, \Psi_l)_{l=1}^L$ be any finite number of local chart for M such that $\cup_{l=1}^L \mathcal{U}_l = M$ and $\Psi_l : \mathcal{U}_l \subset M \to \Psi_l(\mathcal{U}_l) \subset \mathbb{R}^{n-1}$ is homeomorphism. By identifying M with $\{1\} \times M$ and C_1^* with $(0, 1] \times M$, we can then give C_1^* a local chart $(\mathcal{V}_l, \tilde{\Psi}_l)$ by setting $\mathcal{V}_l := (0, 1] \times \mathcal{U}_l$ and $\tilde{\Psi}_l : \mathcal{V}_l = (0, 1] \times \mathcal{U}_l \to \mathbb{R}^n$ with $\tilde{\Psi}_l = id \times \Psi_l$. Note that $\cup_{l=1}^L \mathcal{V}_l = C_1^*$.

We shall use this local chart to define weighted C^k norm and Holder norm for real-valued function on C_1^*. For $k = 0, 1, 2$ and $\alpha \in (0, 1)$, $\nu \in \mathbb{R}$, and $f : C_1^* \to \mathbb{R}$, we define the weighted C^k norm of f as

$$||f||_{C_\nu^k(C_1^*)} := \sup_{0<r\leq 1} \sum_{j=0}^{k} r^{j-\nu} \sup_{[r/2,r]\times M} |\nabla^j f|$$

where the term $\sup_{[r/2,r]\times M} |\nabla^j f|$ is understood in local coordinate as

$$\max_{l=1,\cdots,L} \left(\sup_{|\lambda|=j} \left(\sup_{\tilde{\Psi}_l(\mathcal{V}_l\cap[r/2,r]\times M)} |\partial^\lambda f| \right)\right)$$

here ∂^λ is the size $|\lambda|$ multi-index partial derivative in local coordinate. We then define the weighted C^k space for real-valued function on C_1^* as

$$C_\nu^k(C_1^*) := \left\{ f \in C_{loc}^k(C_1^*) : ||f||_{C_\nu^k(C_1^*)} < \infty \right\}$$

Next we define the weighted Holder seminorm $[f]_{j,\alpha,\nu}$, $j = 0, 1, 2$ as follow:

$$[f]_{j,\alpha,\nu} := \sup_{0<r\leq 1} r^{j+\alpha-\nu} [\nabla^j f]_{\alpha;[r/2,r]\times M}$$

where the term $[\nabla^j f]_{\alpha;[r/2,r]\times M}$ is understood in local coordinate as

$$\max_{l=1,\cdots,L} \left(\sup_{|\lambda|=j} [\partial^\lambda f]_{\alpha;\tilde{\Psi}_l(\mathcal{V}_l\cap[r/2,r]\times M)} \right)$$

here $[.]_{\alpha;\Omega}$ is the usual Holder norm on Euclidean space. We then define weighted Holder space as

$$C^{k,\alpha}_{\nu}(C_1^*) := \left\{ f \in C^{k,\alpha}_{loc}(C_1^*) : ||f||_{C^{k,\alpha}_{\nu}(C_1^*)} < \infty \right\}$$

where

$$||f||_{C^{k,\alpha}_{\nu}} = ||f||_{C^k_{\nu}} + [f]_{k,\alpha,\nu}$$

We now define weighted Holder space for vector-valued maps $C^{k,\alpha}_{\nu}(C_1^*; \mathbb{R}^{n+k})$ which consists of k-times continuous differentiable map $F : C_1^* \to \mathbb{R}^{n+k}$ and each of its component $F = (F^a)_{a=1,\cdots n+k}$ is in $C^{k,\alpha}_{\nu}(C_1^*)$. Moreover, its norm is defined to be

$$||F||_{C^{k,\alpha}_{\nu}} := \sum_a ||F^a||_{C^{k,\alpha}_{\nu}}$$

If the target space is NC, we say that $F \in C^{k,\alpha}_{\nu}(C_1^*; NC)$ if $F \in C^{k,\alpha}_{\nu}(C_1^*; \mathbb{R}^{n+k})$ and $F(x) \in N_x C$ for all $x \in C_1^*$ where NC is the normal bundle of C in $\mathbb{R}^{n+k}$.

3.2 Weighted Schauder Estimate

We now assume that $F \in C^{0,\alpha}_{\nu-2}(C_1^*)$, where $\nu > 3$ and $\Pi_J H \in C^{2,\alpha}(M)$. We hope to get some weighted holder estimate for the solution of the Dirichlet problem in previous section. Precisely we want to find $\Phi \in C^{2,\alpha}_{\nu}(C_1^*)$ satisfying

$$L\Phi = F \text{ on } C_1^*$$
$$\Pi_j \Phi = \Pi_J H \text{ on } M$$

together with the estimate

$$||\Phi||_{C^{2,\alpha}_{\nu}} \le c \left(||F||_{C^{0,\alpha}_{\nu-2}} + ||\Pi_J H||_{C^{2,\alpha}(M)} \right)$$

We will first assume that $\Pi_J H = 0$ and show the following result:

Proposition 3 *Suppose $F \in C^{0,\alpha}_{\nu-2}(C_1^*)$ where $\nu > 3$, $\alpha \in (0, 1)$ is given and $J \ge 1$ satisfies $\Re\gamma_J < \nu < \Re\gamma_{J+1}$ (refer to (8)), then there exist unique solution $\Phi \in C^{2,\alpha}_{\nu}(C_1^*)$ for the equation*

$$L\Phi = F \text{ on } C_1^* \tag{12}$$
$$\Pi_J \Phi = 0 \text{ on } M \tag{13}$$

Moreover, the solution satisfies the estimate

$$||\Phi||_{C^{2,\alpha}_{\nu}} \le c||F||_{C^{0,\alpha}_{\nu-2}}$$

for some constant c independent of Φ, F.

The existence of L^2 unique solution of this Dirichlet problem has been solved in previous section, we only need to estimate the solution in weighted Holder space. We shall first reduce this elliptic system to scalar case and then apply Schauder theory from there. To do so, let $\{e_a\}$, $a = 1, \cdots, n+k$ be standard basis on $\mathbb{R}^{n+k}$, and by taking Euclidean inner product on both side of $L\Phi = F$ with each e_a, we obtain

$$\langle L\Phi, e_a\rangle = \langle F, e_a\rangle =: F^a$$

Recall that $L\Phi = \Delta^\perp\Phi + \sum_{i,j}\langle\Phi, A_{ij}\rangle A_{ij}$ where A_{ij} is second fundamental form of C_1^* with respect to some orthonormal basis $\{\bar{e}_i\}$. Now letting $\{f_i\}$ be orthonormal frame on C we can compute for each a,

$$\begin{aligned}
\langle\Delta^\perp\Phi, e_a\rangle &= \sum_i \langle\nabla^\perp_{f_i}\nabla^\perp_{f_i}\Phi - \nabla^\perp_{(\nabla_{f_i} f_i)^T}\Phi, e_a\rangle \\
&= \sum_i \nabla_{f_i}\langle\nabla_{f_i}\Phi, e_a^\perp\rangle - \langle\nabla^\perp_{f_i}\Phi, \nabla^\perp_{f_i}(e_a^\perp)\rangle - \langle\nabla_{(\nabla_{f_i} f_i)^T}\Phi, e_a^\perp\rangle \\
&= \sum_i \nabla_{f_i}\nabla_{f_i}\langle\Phi, e_a\rangle - \langle\Phi, \nabla^\perp_{f_i}\nabla^\perp_{f_i}(e_a^\perp)\rangle - 2\langle\nabla^\perp_{f_i}\Phi, \nabla^\perp_{f_i}(e_a^\perp)\rangle \\
&\quad - \nabla_{(\nabla_{f_i} f_i)^T}\langle\Phi, e_a\rangle + \langle\Phi, \nabla^\perp_{(\nabla_{f_i} f_i)^T}(e_a^\perp)\rangle \\
&= \Delta\Phi^a - \langle\Phi, \Delta^\perp(e_a^\perp)\rangle - 2\sum_i\langle\nabla^\perp_{f_i}\Phi, \nabla^\perp_{f_i}(e_a^\perp)\rangle
\end{aligned}$$

where $\Phi^a := \langle\Phi, e_a\rangle$ and $\Delta := \sum_i \nabla_{f_i}\nabla_{f_i} - \nabla_{(\nabla_{f_i} f_i)^T}$. Note that we need the fact that Φ is the section of normal bundle in above computation. Since C is minimal in $\mathbb{R}^{n+k}$, we can also write the equation as

$$\begin{aligned}
\Delta\Phi^a :&= \sum_i Hess_{\mathbb{R}^{n+k}}(\Phi^a)(f_i, f_i) \\
&= \sum_i Hess_{\mathbb{R}^{n+k}}(\Phi^a)(f_i, f_i) + d\Phi^a\left(\sum_i \nabla^\perp_{f_i} f_i\right) \\
&= \sum_i Hess_C(\Phi^a)(f_i, f_i) = \Delta_C\Phi^a
\end{aligned}$$

where Δ_C is the Laplace-Beltrami operator on C. Unfortunately this is a couple linear system, for each fix a we cannot directly apply Schauder theory on this equation. However, if we treat the lower order term of Φ as a known term and move it behind, the equation becomes

$$\Delta_C\Phi^a = F^a - G^a(\Phi, \nabla^\perp\Phi) =: Q^a \tag{14}$$

where

$$G^a(\Phi, \nabla^\perp \Phi) := \sum_{i.j} \langle \Phi, A_{ij}\rangle \langle A_{ij}, e_a\rangle - \langle \Phi, \Delta^\perp (e_a^\perp)\rangle - 2\sum_i \langle \nabla^\perp_{f_i} \Phi, \nabla^\perp_{f_i}(e_a^\perp)\rangle$$

Observe that $e_a^\perp \in C_0^{2,\alpha}(C_1^*; NC)$ and

$$\nabla^\perp : \Gamma(TC) \times C_\beta^{k,\alpha}(C_1^*, NC) \to C_{\beta-1}^{k-1,\alpha}(C_1^*, NC)$$

for any integer $k \geq 1$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. So we have

$$\nabla^\perp e_a^\perp \in C_{-1}^{1,\alpha}, \quad \Delta^\perp e_a^\perp \in C_{-2}^{0,\alpha}$$

Using these together with the assumption $\Phi \in C_\nu^{2,\alpha}$ will imply $G^a \in C_{\nu-2}^{0,\alpha}$. To estimate weighted Holder norm of Φ, we need the following proposition which is an application of rescaled Schauder estimate:

Proposition 4 *Let $\Phi \in C_\nu^{2,\alpha}(C_1^*)$ be a solution to the elliptic equation (12). If we assume that*

$$||r^{-\nu}\Phi||_{L^\infty(C_1)} \leq c||r^{2-\nu}F||_{L^\infty(C_1)} \tag{15}$$

where $r = |x|$ is the Euclidean distance to origin and c is constant independent of Φ and F, then we have the estimate

$$||\Phi||_{C_\nu^{2,\alpha}(C_1^*)} \leq c'||F||_{C_{\nu-2}^{0,\alpha}(C_1^*)}$$

where c' is independent of Φ and F.

To prove above proposition, we also need an interpolation inequality of weighted Holder space for scalar function which can be shown by modifying the proof in Lemma 6.32 of [4].

Lemma 5 *Let $f \in C_\nu^{2,\alpha}(B_1^*)$, where $B_1^* = B_1(0)\backslash\{0\}$, $\nu > 0$ then for any $\epsilon > 0$, and $\nu' \leq \nu$, there exist constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that*

$$||f||_{C_{\nu'}^{0,\alpha}} \leq C_1(\epsilon)||r^{-\nu}f||_{L^\infty} + \epsilon||f||_{C_\nu^{2,\alpha}}$$

and

$$||f||_{C_{\nu'}^{1,\alpha}} \leq C_2(\epsilon)||r^{-\nu}f||_{L^\infty} + \epsilon||f||_{C_\nu^{2,\alpha}}$$

Now we shall prove Proposition 4.

Proof of Proposition 4 By taking Euclidean inner product both side of equation (12) with standard basis $\{e^a\}_{a=1}^{n+k}$ of $\mathbb{R}^{n+k}$ we obtain (14). Suppose that $\Phi^a|_M = h^a$, we identify C_1^* with $(0, 1] \times M$ and M with $\{1\} \times M$, then we have an elliptic equation

$$\Delta_C \Phi^a = Q^a \text{ on } (0,1] \times M \tag{16}$$

$$\Phi^a = h^a \text{ on } \{1\} \times M \tag{17}$$

where $h^a(x) = \sum_{j=1}^{J} a_j(1)\langle \phi_j(x), e^a \rangle$ (refer to Sect. 2 for definition of a_j, ϕ_j)

We now choose a local chart for C_1^* in such a way that we can do rescaling estimate in each local chart. For any $p \in M$, using exponential map we can choose a normal neighborhood $\mathcal{U}$ centered at p with radius $r = inj(M)$, where $inj(M)$ is the injectivity radius of M. Since M is compact we can find finitely many geodesic ball $B_{r/2}(p)$ that cover M. Let $\mathcal{U}_l = B_r(p_l)$ and $\Psi_l = exp_{p_l}^{-1} : B_r(p_l) \to \mathbb{R}^{n-1}$ then we have finite number of local chart $(\mathcal{U}_l, \Psi_l)_{l=1,\cdots L}$ for M such that

$$\cup_{l=1}^{L} B_{r/2}(p_l) = M.$$

By setting $\mathcal{V}_l = (0, 1] \times \mathcal{U}_l$ and $\tilde{\Psi}_l = id \times \Psi_l$ for $l = 1, \cdots, L$ we obtain a local chart $(\mathcal{V}_l, \tilde{\Psi}_l)_{l=1}^{L}$ for C_1^*.

Step I: Local estimate

Let $R < 1$ and consider any $p \in C_1^*$ with $|p| \leq R$ i.e. $p = (r_0, y_0) \in (0, 1] \times M$ with $r_0 \leq R$. Using the local chart above, $p \in \mathcal{V}_l$ for some l with local coordinate $(r_0, y(p))$ where

$$y(p) = \Psi_l(y_0) \in \mathbb{R}^{n-1}$$

Let $s = \min\{r_0/3, (1 - r_0)/3, inj(M)/2\}$ and consider a ball $B_s(\tilde{\Psi}_l(p)) \subseteq \tilde{\Psi}_l(\mathcal{V}_l)$. In this local chart, we can perform a rescaling of the function Φ^a by letting

$$v(r, y) := \Phi^a(r_0 + sr, y(p) + sy)$$

defined on $B_1(0) := \{(r, y) : r^2 + |y|^2 \leq 1\} \subset \mathbb{R} \times \mathbb{R}^{n-1}$. Without loss of generality, we can consider a second order elliptic operator L on C_1^* of the form

$$L\Phi^a = a^{ij}\nabla_i\nabla_j\Phi^a + b^i\nabla_i\Phi^a + c\Phi^a$$

where $||a^{ij}||_{C_0^{0,\alpha}}, ||b^i||_{C_{-1}^{0,\alpha}}, ||c||_{C_{-2}^{0,\alpha}} < \infty$.

Under this scaling, v satisfies the equation

$$\bar{L}v = f \text{ on } B_1(0) \tag{18}$$

where

$$\bar{L} = \bar{a}^{ij}\partial_i\partial_j + s\bar{b}^i\partial_i + s^2\bar{c}$$

is second order elliptic linear operator on $B_1(0)$ with Holder continuous coefficient and

$$f(r, y) = s^2 Q^a(r_0 + sr, y(p) + sy).$$

By standard elliptic interior estimate

$$||v||_{2,\alpha;B_{1/2}(0)} \leq c(||v||_{0;B_1(0)} + ||f||_{0,\alpha;B_1(0)}) \tag{19}$$

Claim 1:

$$||f||_{0,\alpha;B_1(0)} \leq cs^\nu ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \tag{20}$$

where c depends only on R and $inj(M)$.
Proof of Claim 1: For $t_0 = \min\{4r_0/3, 1\}$, we define $A_{t_0} := [t_0/2, t_0] \times M$. Observe that in local chart, $B_s(\tilde{\Psi}_l(p)) \subseteq [t_0/2, t_0] \times \Psi_l(\mathcal{U}_l)$ where $s = \min\{r_0/3, (1 - r_0)/3, inj(M)/2\}$, then by scaling backward,

$$\begin{aligned}
\sup_{B_1(0)} |f| &= s^2 \sup_{B_s(\tilde{\Psi}_l(p))} |Q^a| \\
&\leq s^2 \sup_{[t_0/2,t_0]\times\Psi_l(\mathcal{U}_l)} |Q^a| \\
&\leq s^2 \sup_{A_{t_0}} t_0^{\nu-2} |t_0^{2-\nu} Q^a(x)| \\
&= s^\nu \sup_{A_{t_0}} (t_0/s)^{\nu-2} |t_0^{2-\nu} Q^a|
\end{aligned}$$

if $t_0 = 4r_0/3$ $(r_0 \leq 3/4)$ then by definition of s,

$$\frac{t_0}{s} \leq \max\left\{4, \frac{4R}{1-R}, \frac{2}{inj(M)}\right\}$$

if $t_0 = 1$ $(r_0 > 3/4)$ then

$$\frac{t_0}{s} \leq \max\left\{4, \frac{3}{1-R}, \frac{2}{inj(M)}\right\}$$

In both case, we have an upper bound for t_0/s depending only on R and $inj(M)$, so we have

$$\sup_{B_1(0)} |f| \leq c_1 s^\nu ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \tag{21}$$

where c_1 only depends on R and $inj(M)$. We can similarly estimate the Holder norm,

$$[f]_{0,\alpha;B_1(0)} = s^{2+\alpha}[Q^a]_{0,\alpha;B_s(\tilde{\Psi}_l(p))}$$

$$
\begin{aligned}
&\le s^{2+\alpha} t_0^{\nu-2-\alpha} t_0^{2-\nu+\alpha} [Q^a]_{0,\alpha;A_{t_0}} \\
&= s^{\nu} (t_0/s)^{\nu-2-\alpha} (t_0^{2-\nu+\alpha} [Q^a]_{0,\alpha;A_{t_0}}) \\
&\le c_2 s^{\nu} ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)}
\end{aligned}
$$

where c_2 only depend on R and $inj(M)$. Combine both estimate we prove Claim 1.
Claim 2:

$$||v||_{0;B_1(0)} \le cs^{\nu} ||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \tag{22}$$

where c is independent of Φ and F.
Proof of Claim 2: By scaling backward and note that $B_s(\tilde{\Psi}_l(p)) \subseteq [r_0 - s, r_0 + s] \times \Psi_l(\mathcal{U}_l)$,

$$
\begin{aligned}
||v||_{0;B_1(0)} &= \sup_{B_s(\tilde{\Psi}_l(p))} |\Phi^a| \\
&\le \sup_{[r_0-s,r_0+s]\times\Psi_l(\mathcal{U}_l)} |\Phi^a| \\
&\le \sup_{[r_0-s,r_0+s]\times M} |\Phi^a| \\
&= s^{\nu} \sup_{x\in[r_0-s,r_0+s]\times M} (r_x/s)^{\nu} |r_x^{-\nu} \Phi^a|
\end{aligned}
$$

where $r_x = dist(x, 0)$ the Euclidean distance of x to 0. Since we know r_x satisfies

$$r_0 - s \le r_x \le r_0 + s$$

we can then estimate

$$\frac{r_x}{s} \le 1 + \frac{r_0}{s} \le \max\left\{4, \frac{1+2R}{1-R}, 1 + \frac{2}{inj(M)}\right\}$$

So

$$
\begin{aligned}
||v||_{0;B_1(0)} &\le c_3 s^{\nu} ||r^{-\nu} \Phi^a||_{L^\infty(C_1)} \\
&\le c_4 s^{\nu} ||s^{2-\nu} F||_{L^\infty(C_1)} \quad \text{by assumption (15)} \\
&\le c_4 s^{\nu} ||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)}
\end{aligned}
$$

for some constant c_4 independent of Φ, F. Hence we prove Claim 2.

Using (20), (22), Schauder interior estimate (19) and perform rescaling backward, we obtain a local estimate in a local chart

$$||v||_{2,\alpha;B_{1/2}(0)} = \sum_{j=0}^{2} s^j \sup_{B_{s/2}(\tilde{\Psi}_l(p))} |\nabla^j \Phi^a| + s^{2+\alpha} [\nabla^2 \Phi^a]_{\alpha;B_{s/2}(\tilde{\Psi}_l(p)))} \tag{23}$$

$$\leq c_R s^{\nu} \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \tag{24}$$

where c_R is independent of Φ and F.
Step II: Interior estimate
For any $t \in (0, R]$, let $A_t := [t/2, t] \times M$, and suppose the supremum of $\sum_k t^{k-\nu} |\nabla^k \Phi^a|$ on A_t occurs at $p' \in A_t$. Without loss of generality, suppose p' lies in the chart $(\mathcal{V}, \tilde{\Psi})$, then $B_{s/2}\tilde{\Psi}(p') \subseteq \tilde{\Psi}(\mathcal{V})$ where $s = \min\{|p'|/3, (1 - |p'|)/3, inj(M)/2\}$. We first estimate the ratio of s/t as follows:

$$p' \in A_t \iff \frac{t}{2} \leq |p'| \leq t \iff \frac{1}{2} \leq \frac{|p'|}{t} \leq 1$$

hence

$$\frac{s}{t} \leq \frac{|p'|}{3t} \leq \frac{1}{3}$$

Now using local estimate in Step I, we have

$$\sum_{j=0}^{2} t^{j-\nu} \sup_{A_t} |\nabla^j \Phi^a| = \sum_{j=0}^{2} t^{j-\nu} \sup_{B_{s/2}\tilde{\Psi}(p')} |\nabla^j \Phi^a| \tag{25}$$

$$= \sum_{j=0}^{2} (s/t)^{\nu-j} s^{-\nu} s^j \sup_{B_{s/2}\tilde{\Psi}(p')} |\nabla^j \Phi^a| \tag{26}$$

$$\leq \sum_{j=0}^{2} \frac{1}{3^{\nu-k}} s^{-\nu} s^j \sup_{B_{s/2}\tilde{\Psi}(p')} |\nabla^j \Phi^a| \tag{27}$$

$$\leq c_R' \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \tag{28}$$

where we use (24) in the last inequality. To estimate Holder norm, we use the following estimate for any bounded $\Omega \subset \mathbb{R}^n$, and $\eta \in C^{0,\alpha}(\Omega)$,

$$[\eta]_{\alpha;\Omega} \leq 2^{1-\alpha} \sup_{p\in\Omega} ([\eta]_{\alpha;B_{s/2}(p)\cap\Omega} + s^{-\alpha} |\eta|_{0;B_{s/2}(p)\cap\Omega}) \tag{29}$$

which can be proved by considering two cases either $y \in B_{s/2}(x) \cap \Omega$ or $y \notin B_{s/2}(x) \cap \Omega$.

On any local chart $(\mathcal{V}, \tilde{\Psi})$, using (29) with $\Omega = [t/2, t] \times \tilde{\Psi}(\mathcal{V})$ and $B_{s/2}(p) \subseteq \Omega$ with $p \in \Omega$ and $s = s(p)$ is chosen similarly as previous,

$$\begin{aligned} t^{2+\alpha-\nu} [\nabla^2 \Phi^a]_{\alpha;\Omega} &\leq 2^{1-\alpha} t^{2+\alpha-\nu} \sup_{p\in\Omega} \left([\nabla^2 \Phi^a]_{\alpha;B_{s/2}(p)} + s^{-\alpha} |\nabla^2 \Phi^a|_{C^0(B_{s/2}(p))} \right) \\ &= 2^{1-\alpha} (s/t)^{\nu-2-\alpha} s^{-\nu} \sup_{p\in\Omega} \left(s^{2+\alpha} [\nabla^2 \Phi^a]_{\alpha;B_{s/2}(p)} + s^2 |\Phi^a|_{0;B_{s/2}(p)} \right) \\ &\leq c_R' \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \end{aligned}$$

where c'_R is independent of Φ and F. Combine both estimate we obtain the following interior estimate

$$||\Phi^a||_{C^{2,\alpha}_{\nu}((0,R]\times M)} \leq c_R \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \tag{30}$$

Step III: Boundary estimate
Using boundary Schauder estimate in [4,Lemma 6.5], there exist $\delta > 1/2$ such that on $\Omega_\delta := [\delta, 1] \times M$,

$$||\Phi^a||_{C^{2,\alpha}(\Omega_\delta)} \leq c_5 \left(||\Phi^a||_{C^0(C_1^*)} + ||Q^a||_{C^{0,\alpha}(C_1^*)} + ||h^a||_{C^{2,\alpha}(M)} \right) \tag{31}$$

Also we have

$$||\Phi^a||_{C^0(C_1^*)} \leq ||r^{-\nu}\Phi^a||_\infty \leq ||r^{2-\nu}F||_\infty \leq ||F||_{C^{0,\alpha}_{\nu-2}} \tag{32}$$

$$||Q^a||_{C^{0,\alpha}(C_1^*)} \leq ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \tag{33}$$

Now we want to estimate $||h||$ in term of F. Recall that $h^a(x) = \sum_{j=1}^{J} a_j(1)\langle \phi_j(x), e^a \rangle$ and for $1 \leq j \leq J$, $0 < t \leq 1$ we have

$$|a_j(t)| \leq ct^\nu ||r^{2-\nu}F||_\infty$$

consequently

$$||h^a||_{C^{2,\alpha}(M)} \leq \sum_{j=1}^{J} |a_j(1)|||\phi_j||_{C^{2,\alpha}(M)} \leq C||F||_{C^{0,\alpha}_{\nu-2}} \tag{34}$$

where $C = C(J, ||\phi_j||_{2,\alpha}, n, \alpha, \nu)$. Putting everything together we obtain boundary Schauder estimate

$$||\Phi^a||_{C^{2,\alpha}(\Omega_\delta)} \leq c_6 \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \tag{35}$$

By choosing $R = (1 + \delta)/2$ in the interior estimate and together with boundary estimate, we obtain the following global estimate:

$$||\Phi^a||_{C^{2,\alpha}_{\nu}(C_1^*)} \leq c_7 \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||Q^a||_{C^{0,\alpha}_{\nu-2}(C_1^*)} \right) \tag{36}$$

Step IV: Interpolation
Recall $Q^a = F^a - G^a$ from (14) and we have

$$||Q^a||_{C^{0,\alpha}_{\nu-2}} \leq ||F||_{C^{0,\alpha}_{\nu-2}} + ||G^a||_{C^{0,\alpha}_{\nu-2}}$$

from definition of G^a, we have

$$||G^a||_{C^{0,\alpha}_{\nu-2}} \leq c_8 \left(||\Phi||_{C^{0,\alpha}_{\nu}} + ||\nabla \Phi||_{C^{0,\alpha}_{\nu-1}} \right) \leq c_8 \left(||\Phi||_{C^{0,\alpha}_{\nu}} + ||\Phi||_{C^{1,\alpha}_{\nu}} \right)$$

Plugging this back into the global estimate (36) and taking sum both side for $a = 1, \cdots, n+k$,

$$||\Phi||_{C^{2,\alpha}_{\nu}(C_1^*)} \leq c_9 \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + ||\Phi||_{C^{0,\alpha}_{\nu}(C_1^*)} + ||\Phi||_{C^{1,\alpha}_{\nu}(C_1^*)} \right)$$

By interpolation inequality in Lemma 5, pick $\epsilon > 0$ satisfying $\epsilon c_9 < 1/2$, there exist $C(\epsilon) > 0$ such that

$$||\Phi||_{C^{2,\alpha}_{\nu}(C_1^*)} \leq c_9 \left(||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + \epsilon ||\Phi||_{C^{2,\alpha}_{\nu}} + C(\epsilon) ||r^{-\nu}\Phi||_{\infty} \right) \tag{37}$$

$$\leq c_9 \left((1 + C(\epsilon)) ||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)} + \epsilon ||\Phi||_{C^{2,\alpha}_{\nu}} \right) \tag{38}$$

Hence we prove the estimate

$$||\Phi||_{C^{2,\alpha}_{\nu}(C_1^*)} \leq c_{10} ||F||_{C^{0,\alpha}_{\nu-2}(C_1^*)}$$

and complete the proof of Proposition 4. □

3.3 Weighted C^0 Estimate

The implication of Proposition 4 is that knowing the weighted sup norm of the solution in term of weighted sup norm of the source term is suffice to estimate the weighted Holder norm. For this, we reduce the estimate of weighted Holder norm of Φ to estimate its weighted sup norm instead.

The first observation is the norm $||\Phi||_{C^0_{\nu}(C_1^*)}$ ie equivalent to $||r^{-\nu}\Phi||_{L^{\infty}(C_1)}$ when $\nu > 0$. Indeed, for any $x \in A_r := [r/2, r] \times M$, we have

$$\frac{1}{2^{\nu}} |r_x^{-\nu} \Phi(x)| \leq r^{-\nu} |\Phi(x)| = (r/r_x)^{-\nu} |r_x^{-\nu} \Phi(x)| \leq |r_x^{-\nu} \Phi(x)|$$

where $r_x = |x|$. So we can interchange between these two equivalent norms while doing estimate. The equivalent norm is also true for any $\nu \in \mathbb{R}$ but we don't need it here.

In this section, we always assume that $F \in C^{0,\alpha}_{\nu-2}(C_1^*)$ for $\nu > 3$ and $\Phi \in C^{2,\alpha}_{loc}(C_1^*)$ is the unique solution given by Theorem 2, i.e.

$$\Delta^{\perp}\Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij} = F \quad \text{on } C_1^*$$

and

$$\Pi_J \Phi = 0 \text{ on } M.$$

Let $u(x) := |\Phi(x)|$ for $x \in C_1^*$, we have the following local estimate :

Lemma 6 *Suppose $B_{3R}(x) \subseteq C_1^*$ is a geodesic ball of radius $3R$, then there exist a constant $c(q) > 0$ with $q > n$, $n = dimC$ such that*

$$|u|_{0;B_R(x)} \leq c(q) \left(R^{-n/2} |u|_{L^2(B_{2R}(x))} + R^{2(1-n/q)} |F|_{L^{q/2}(B_{2R}(x))} \right) \tag{39}$$

Proof We first show that u satisfies the following inequality in $(0, 1) \times M$ (in distributional sense)

$$\Delta u \geq -|A|^2 u - |F|$$

Regularize u by considering $u_\delta := \sqrt{u^2 + \delta}$ for any $\delta > 0$. Choose a local orthonormal frame $\{e_i\}$ then we can compute

$$\Delta u_\delta = \sum_i e_i e_i u_\delta - (\nabla_{e_i} e_i) u_\delta \tag{40}$$

$$= u_\delta^{-1} \langle \Delta^\perp \Phi, \Phi \rangle + u_\delta^{-1} |\nabla \Phi|^2 - \sum_i u_\delta^{-3} \langle \nabla_{e_i} \Phi, \Phi \rangle^2 \tag{41}$$

$$= u_\delta^{-1} \left(\langle F, \Phi \rangle - \sum_{i,j} \langle \Phi, A_{ij} \rangle^2 \right) + u_\delta^{-1} |\nabla \Phi|^2 - \sum_i u_\delta^{-3} \langle \nabla_{e_i} \Phi, \Phi \rangle^2 \tag{42}$$

$$\geq -u_\delta^{-1} \left(|F||\Phi| + |A|^2 |\Phi|^2 \right) \geq -|F| - |A|^2 u_\delta \tag{43}$$

For any $\eta \in C_0^1((0, 1) \times M)$ with $\eta \geq 0$, we then have

$$\int_{(0,1)\times M} \frac{1}{2u_\delta} \langle \nabla \eta, \nabla (u^2) \rangle = \int_{(0,1)\times M} \langle \nabla \eta, \nabla u_\delta \rangle \leq \int_{(0,1)\times M} |F|\eta + |A|^2 \eta u_\delta$$

Let $\delta \to 0$ and use Fatou Lemma, we can conclude that u is a subsolution of the following elliptic equation(in distributional sense):

$$\Delta w + |A|^2 w = -|F| \tag{44}$$

We can then apply the C^0 estimate from [4,Theorem 8.17] with $p = 2$ to obtain the interior estimate. □

Next we want to estimate the weighted C^0 norm of u near the origin. Recall $A_r := [r/2, r] \times M$, and we shall show the following estimate:

Proposition 7 *For* $0 < r < 2/3$, u *satisfies*

$$r^{-\nu}|u|_{0;A_r} \leq C||r^{2-\nu}F||_{L^\infty(C_1)}$$

for some constant C *independent of* u *and* F.

Proof By choosing $R = r/6$, then for any $x \in A_r$ we have

$$B_{3R}(x) \subseteq C_1^* \quad \text{and} \quad B_{2R}(x) \subseteq [r/6, 4r/3] \times M$$

We can then cover A_r by finitely many $B_R(x)$ and apply Lemma 6 with $q = 2n$ in each $B_R(x)$ to obtain estimate on A_r as follows:

$$|u|_{0;A_r} \leq c_{10}\left(r^{-n/2}|u|_{L^2([r/6,4r/3]\times M)} + r|F|_{L^n([r/6,4r/3]\times M)}\right) \tag{45}$$

We first compute the term $|F|_{L^n([r/6,4r/3]\times M)}$,

$$\begin{aligned}
|F|^n_{L^n([r/6,4r/3]\times M)} &= \int_{r/6}^{4r/3}\int_M |F|^n s^{n-1} dv_M ds \\
&= \int_M \int_{r/6}^{4r/3} s^{n-1} s^{n\nu-2n}|s^{2-\nu}F|^n ds dv_M \\
&\leq ||r^{2-\nu}F||^n_{L^\infty([r/6,4r/3]\times M)} \int_M \int_{r/6}^{4r/3} s^{n\nu-n-1} ds dv_M \\
&\leq c_{11}||r^{2-\nu}F||^n_{L^\infty(C_1^*)} r^{n(\nu-1)}
\end{aligned}$$

so we have

$$r|F|_{L^n([r/6,4r/3]\times M)} \leq \sqrt[n]{c_{11}}\, r^\nu ||r^{2-\nu}F||_{L^\infty(C_1^*)} \tag{46}$$

Next we compute the term $|u|_{L^2([r/6,4r/3]\times M)}$,

$$\begin{aligned}
|u|^2_{L^2([r/6,4r/3]\times M)} &= \int_{r/6}^{4r/3} s^{n-1} \int_M u^2 dv_M ds \\
&= \int_{r/6}^{4r/3} s^{n-1}||\Phi(s,.)||^2_{L^2(M)} ds \\
&\leq c_{12}||r^{2-\nu}F||^2_{L^\infty(C_1)} \int_{r/6}^{4r/3} s^{n-1}s^{2\nu} ds \qquad \text{from (11)} \\
&\leq c_{13} r^{2\nu+n}||r^{2-\nu}F||^2_{L^\infty(C_1)}
\end{aligned}$$

This implies

$$r^{-n/2}|u|_{L^2([r/6,4r/3]\times M)} \leq \sqrt{c_{13}}\, r^\nu ||r^{2-\nu}F||^2_{L^\infty(C_1)} \tag{47}$$

By taking a bigger constant of (46) and (47), we have from (45)

$$|u|_{0;A_r} \leq c_{14} r^{\nu} ||r^{2-\nu} F||_{L^\infty(C_1)} \tag{48}$$

This complete the proof of the proposition. □

Finally we are in the position to show global weighted C^0 estimate of Φ.

Proposition 8 *Let $\Phi \in C^{2,\alpha}_{loc}(C_1^*)$ be the solution given by Theorem 2 then Φ satisfies the estimate*

$$||r^{-\nu}\Phi||_{L^\infty(C_1)} \leq C||r^{2-\nu}F||_{L^\infty(C_1)}$$

for some constant $C > 0$ independent of Φ and F.

Proof As before, let $u := |\Phi|$ and we know from (44) that u satisfies

$$\Delta u \geq -|A|^2 u - |F| = -|A^M|^2 u/|x|^2 - |F|$$

on $(0, 1) \times M$ where A^M is second fundamental form of M in $\mathbb{S}^{n+k-1}$. For any fix $0 < R_0 < 1$, we can then apply the global C^0 estimate of [4,Theorem 8.16] with $q = 2n$ to the equation

$$\Delta w = f := -|A^M|^2 u/|x|^2 - |F|$$

on $[R_0, 1] \times M$ and obtain

$$|u|_{0;[R_0,1]\times M} \leq \sup_{\{1\}\times M} u + \sup_{\{R_0\}\times M} u + c_{15}||f||_{L^n([R_0,1]\times M)} \tag{49}$$

Since $\Pi_J \Phi = 0$ on $\{1\} \times M$ and $u|_M = |\Phi(1, .)|$, we have

$$|\Phi(1, .)| \leq \sum_{j\leq J} |a_j(1)||\phi_j| \leq c_{16}||r^{2-\nu}F||_\infty \tag{50}$$

If we choose $R_0 < 2/3$, we can use Proposition 7 to claim that

$$\sup_{\{R_0\}\times M} u \leq C R_0^{\nu}||r^{2-\nu}F||_\infty \tag{51}$$

To estimate L^n norm of f, we need to estimate both L^n norm of $|F|$ and $u/|x|^2$. By just changing the limit of integral, our previous L^n norm estimate of $|F|$ gives

$$|F|_{L^n([R_0,1]\times M)} \leq c_{12}||r^{2-\nu}F||_\infty \tag{52}$$

So we only need to estimate L^n norm of $u/|x|^2$. One observation is the L^n norm with $n > 2$ can be estimated in term of C^0 and L^2 norm. To be precise, for any $w \in L^2$ and any $\epsilon > 0$, apply Young inequality with $(p, q) = (\frac{n}{n-2}, \frac{n}{2})$ to obtain

$$\begin{aligned}\left(\int |w|^n dx\right)^{1/n} &= \left(\int |w|^{n-2}|w|^2 dx\right)^{1/n} \\ &\leq |w|_0^{(n-2)/n}|w|_{L^2}^{2/n} \\ &\leq \frac{\epsilon^p}{p}|w|_0 + \frac{1}{q\epsilon^q}|w|_{L^2}\end{aligned}$$

By replacing w with $u/|x|^2$ in above inequality, we obtain

$$\begin{aligned}||\frac{u}{|x|^2}||_{L^n([R_0,1]\times M)} &\leq \frac{\epsilon^p}{p} \sup_{[R_0,1]\times M} \frac{u}{|x|^2} + \frac{1}{q\epsilon^q}||\frac{u}{|x|^2}||_{L^2([R_0,1]\times M)} \\ &\leq \frac{\epsilon^p}{pR_0^2} \sup_{[R_0,1]\times M} u + \frac{1}{q\epsilon^q}\left(\int_{R_0}^1 \int_M (\frac{u}{s^2})^2 s^{n-1} dv_M ds\right)^{1/2} \\ &\leq \frac{\epsilon^p}{pR_0^2} \sup_{[R_0,1]\times M} u + \frac{c_{14}}{q\epsilon^q}||r^{2-\nu}F||_\infty \left(\int_{R_0}^1 s^{n+2\nu-5} ds\right)^{1/2} \\ &= \frac{\epsilon^p}{pR_0^2} \sup_{[R_0,1]\times M} u + \frac{c_{14}}{q\epsilon^q}\sqrt{\frac{1-R_0^{n+2\nu-4}}{n+2\nu-4}}||r^{2-\nu}F||_\infty\end{aligned}$$

Combining above estimate with (50), (51), (52) then the global estimate in (49) becomes

$$\begin{aligned}|u|_{0;[R_0,1]\times M} \leq{}& ||r^{2-\nu}F||_\infty \left(c_{16} + CR_0^\nu\right) + c_{15}[c_{12}||r^{2-\nu}F||_\infty \\ &+ \sup_M |A^M|^2 \left(\frac{\epsilon^p}{pR_0^2} \sup_{[R_0,1]\times M} u + \frac{c_{14}}{q\epsilon^q}\sqrt{\frac{1-R_0^{n+2\nu-4}}{n+2\nu-4}}||r^{2-\nu}F||_\infty\right)\Bigg]\end{aligned}$$

Now we can choose $\epsilon > 0$ such that it satisfies

$$c_{15} \sup_M |A^M|^2 \frac{\epsilon^p}{pR_0^2} = \frac{1}{2}$$

then we can conclude that

$$|u|_{0;[R_0,1]\times M} \leq c_{17}||r^{2-\nu}F||_{L^\infty(C_1^*)} \tag{53}$$

To prove the global estimate on C_1^*, by the equivalent norm of $||r^{-\nu}\Phi||_\infty$ and $||\Phi||_{C_\nu^0}$, it is suffice to show

$$r^{-\nu} \sup_{A_r} u \leq C||r^{2-\nu}F||_{L^\infty(C_1^*)}$$

for any $0 < r < 1$.

When $0 < r < 2/3$, this is just the result in Proposition 7.

When $1 > r > 2/3$, if $x \in A_r \cap [R_0, 1] \times M$, then by (53)

$$r^{-\nu} u(x) \leq c_{17} (2/3)^{-\nu} ||r^{2-\nu} F||_\infty$$

otherwise, we have $x \in A_r \cap \{|x| < R_0\}$ and we can apply Proposition 7 again to conclude the result. Hence we complete the proof. □

We shall now restate Proposition 3 which is an immediate consequence of Proposition 8 and Proposition 4.

Proposition 3 Suppose $F \in C^{0,\alpha}_{\nu-2}(C_1^*)$ where $\nu > 3$, $\alpha \in (0, 1)$ is given and $J \geq 1$ satisfies $\gamma_J < \nu < \gamma_{J+1}$ (refer to (8)), then there exist solution $\Phi \in C^{2,\alpha}_\nu(C_1^*)$ for the equation

$$\begin{aligned} L\Phi &= F \text{ on } C_1^* \\ \Pi_J \Phi &= 0 \text{ on } M \end{aligned}$$

Moreover, the solution satisfies the estimate

$$||\Phi||_{C^{2,\alpha}_\nu} \leq c||F||_{C^{0,\alpha}_{\nu-2}}$$

for some constant c independent of Φ, F.

For general $C^{2,\alpha}$ boundary data, we can show the following result:

Theorem 9 *Suppose $F \in C^{0,\alpha}_{\nu-2}(C_1^*)$ where $\nu > 3$, $\alpha \in (0, 1)$ is given and $J \geq 1$ satisfies $\Re\gamma_J < \nu < \Re\gamma_{J+1}$ (refer to equation 8). Also given a map $H \in C^{2,\alpha}(M)$, there exist a unique solution $\Phi \in C^{2,\alpha}_\nu(C_1^*)$ to the equation*

$$L\Phi = F \text{ on } C_1^* \tag{54}$$

$$\Pi_J \Phi = \Pi_J H \text{ on } M \tag{55}$$

together with the estimate

$$||\Phi||_{C^{2,\alpha}_\nu} \leq K_1 \left(||F||_{C^{0,\alpha}_{\nu-2}} + ||\Pi_J H||_{C^{2,\alpha}(M)} \right)$$

where K_1 is independent of Φ, F, H.

Proof Let $\hat{\Pi_J} H : C_1 \to \mathbb{R}^{n+k}$ be $C^{2,\alpha}$ extension of $\Pi_J H$ (see for example [4,Lemma 6.38]) with

$$||\hat{\Pi_J} H||_{C^{2,\alpha}(C_1^*)} \leq c||\Pi_J H||_{C^{2,\alpha}(M)}$$

where the constant c is independent of H.

Let $V = \Phi - r^{\nu}\Pi_{\hat{J}}H$, where $r = |x|$, then we have

$$LV = F - L(r^{\nu}\Pi_{\hat{J}}H) \quad \text{on } C_1^*$$

and $\Pi_J V = 0$ on M. Also note that $r^{\nu}\Pi_{\hat{J}}H \in C_{\nu}^{2,\alpha}(C_1^*)$ with estimate

$$||r^{\nu}\Pi_{\hat{J}}H||_{C_{\nu}^{2,\alpha}(C_1^*)} \leq C||\Pi_{\hat{J}}H||_{C^{2,\alpha}(C_1^*)})$$

By applying Proposition 3 for V,

$$\begin{aligned}
||\Phi||_{C_{\nu}^{2,\alpha}} &\leq ||V||_{C_{\nu}^{2,\alpha}} + ||r^{\nu}\Pi_{\hat{J}}H||_{C_{\nu}^{2,\alpha}} \\
&\leq c(||F||_{C_{\nu-2}^{0,\alpha}} + ||Lr^{\nu}\Pi_{\hat{J}}H||_{C_{\nu-2}^{0,\alpha}}) + ||r^{\nu}\Pi_{\hat{J}}H||_{C_{\nu}^{2,\alpha}} \\
&\leq c'(||F||_{C_{\nu-2}^{0,\alpha}} + ||\Pi_{\hat{J}}H||_{C^{2,\alpha}(C_1^*)}) \\
&\leq c''(||F||_{C_{\nu-2}^{0,\alpha}} + ||\Pi_J H||_{C^{2,\alpha(M)}})
\end{aligned}$$

□

We have used the following estimate in the third inequality

$$||L(r^{\nu}\Pi_{\hat{J}}H)||_{C_{\nu-2}^{0,\alpha}} \leq c||\Pi_{\hat{J}}H||_{C^{2,\alpha}(C_1^*)}$$

4 Solving Nonlinear Equation

We are now ready to show the existence of solution to $M_G(\Phi) = 0$ with decay $O(|x|^{\nu})$ at the origin. For convenient, write $\mathcal{B} := C_{\nu}^{2,\alpha}(C_1^*; NC)$ where $\alpha \in (0, 1)$ and $\nu > 3$.

Given $H \in C^{2,\alpha}(M)$, we consider an operator $\mathcal{U} : \mathcal{B} \to \mathcal{B}$ that maps $W \in \mathcal{B}$ to the solution $\Phi = \mathcal{U}(W)$ given by Theorem 2 :

$$\begin{aligned}
L\Phi &= -Q(W/r, \nabla^{\perp}W, (\nabla^{\perp})^2 W) =: F(W) && \text{on } C_1^* \\
\Pi_J\Phi &= \Pi_J H && \text{on } M
\end{aligned}$$

The definition of Q above can be referred from equation (2). Let us recall the contraction mapping principle:

Theorem 10 *[4,Theorem 5.1] Let T be a contraction mapping from a Banach space $\mathfrak{B}$ into itself, i.e. there exist a number $\theta < 1$ such that*

$$\|Tx - Ty\|_{\mathfrak{B}} \leq \theta\|x - y\|_{\mathfrak{B}}, \qquad \forall x, y \in \mathfrak{B}$$

Then T has a unique fixed point, that is there exist a unique $x \in \mathfrak{B}$ such that $Tx = x$.

In order to solve the nonlinear problem $M_G(\Phi) = 0$, we reformulate the problem to finding the fix point of the operator $\mathcal{U}$ above. To do so, we choose a closed subset of $\mathcal{B}$ as follow:

Define $\mathcal{K} \subset \mathcal{B}$ to be the set

$$\mathcal{K} := \{W \in \mathcal{B} : ||W||_{\mathcal{B}} \le A||\Pi_J H||_{C^{2,\alpha}}\}$$

where constant A to be determined later. Once we show that $\mathcal{U}$ is a contraction maps from $\mathcal{K}$ into $\mathcal{K}$, we can apply contraction mapping principle above to conclude the existence of fix point for $\mathcal{U}$. To this end, it is suffice to show the following lemma:

Lemma 11 *Let $W_i \in \mathcal{B}$, $i = 1, 2$ with $||W_i||_{\mathcal{B}} < 1$, then $F(W_i) \in C^{0,\alpha}_{\nu-2}(C_1^*)$ and $F(W_i)$ satisfy the inequality*

$$||F(W_1)||_{C^{0,\alpha}_{\nu-2}} \le K_2||W_1||^2_{\mathcal{B}}$$

and

$$||F(W_1) - F(W_2)||_{C^{0,\alpha}_{\nu-2}} \le K_3(||W_1||_{\mathcal{B}} + ||W_2||_{\mathcal{B}})||W_1 - W_2||_{\mathcal{B}}$$

for some positive constants K_2, K_3 independent of W_1, W_2.

Proof We first recall that $Q(w, z, p)$ is a polynomial of w, z, p which is at least degree 2 and is linear in p where $w \in \mathbb{R}^N$, $z \in \mathbb{R}^{nN}$, $p \in \mathbb{R}^{n^2N}$ with $N = n + k$. Moreover, when $|w/r| < 1$ and $|z| < 1$, we have (see (3)) for some constant μ,

$$|Q(w/r, z, p)| \le \mu\left(\frac{1}{r}\left(|\frac{w}{r}|^2 + |z|^2\right) + \left(|\frac{w}{r}| + |z|\right)|p|\right)$$

To prove the first inequality, we observe that for $A_r = [r/2, r] \times M$, (we shall omit the subscript i of W_i) by using the definition of weighted norm of $C^{2,\alpha}_\nu$ in Sect. 3.1, the sup norm and Holder norm in A_r can be estimated as follow:

i. $|W/r|_0 \le r^{\nu-1}|W|_{\mathcal{B}}$
ii. $|\nabla W|_0 \le r^{\nu-1}|W|_{\mathcal{B}}$
iii. $|\nabla^2 W|_0 \le r^{\nu-2}|W|_{\mathcal{B}}$
iv. $[W/r]_\alpha \le r^{\nu-\alpha-1}|W|_{\mathcal{B}}$
v. $[\nabla W]_\alpha \le r^{\nu-\alpha-1}|W|_{\mathcal{B}}$
vi. $[\nabla^2 W]_\alpha \le r^{\nu-\alpha-2}|W|_{\mathcal{B}}$

Hence the sup norm and Holder norm of $Q(W/r, \nabla W, \nabla^2 W)$ in A_r can be estimated as follow:

$$|Q|_0 \le cr^{2\nu-3}|W|^2_{\mathcal{B}}$$

and using the fact that $[fg]_\alpha \le |f|_0[g]_\alpha + [f]_\alpha|g|_0$ we obtain

$$[Q]_\alpha \le c'(1/r|W/r|_0[W/r]_\alpha + 1/r|\nabla W|_0[\nabla W]_\alpha + |W/r|_0[\nabla^2 W]_\alpha$$

$$
\begin{aligned}
&+ [W/r]_\alpha |\nabla^2 W|_0 + |\nabla W|_0 [\nabla^2 W]_\alpha + [\nabla W]_\alpha |\nabla^2 W|_0) \\
&\le c'' r^{2\nu-3-\alpha} |W|_{\mathcal{B}}^2
\end{aligned}
$$

Combining these two estimate we have in A_r, $0 < r < 1$,

$$
r^{-\nu+2}|Q|_0 + r^{-\nu+2+\alpha}[Q]_\alpha \le C r^{\nu-1}|W|_{\mathcal{B}}^2 \le C|W|_{\mathcal{B}}^2
$$

which is the estimate for $||F(W)||_{C^{0,\alpha}_{\nu-2}}$. So we have proven the first inequality.

For the second inequality, we write

$$
F(W_1) - F(W_2) = \int_0^1 \frac{d}{dt} F(W(t))dt
$$

where $W(t) = tW_1 + (1-t)W_2$. Then we have

$$
\left|\frac{d}{dt}F(W(t))\right| \le \left|\langle D_{w_a} Q, \dot{W}^a(t)/r\rangle\right| + \left|\langle D_{z_{ai}} Q, \nabla_i \dot{W}^a(t)\rangle\right| + \left|\langle D_{p_{aij}} Q, \nabla^2_{i,j} \dot{W}^a(t)\rangle\right|
$$

where $\dot{W}$ means the derivative of $W(t)$ w.r.t t and we are summing up the index of right hand side with $a = 1, \cdots N$ and $i, j = 1, \cdots n$.

By property of Q (see (3)), we have

$$
|D_w Q| \le \mu(1/r(|W(t)|/r + |\nabla W|) + |\nabla^2 W(t)|) \tag{56}
$$

$$
|D_z Q| \le \mu(1/r(|W(t)|/r + |\nabla W|) + |\nabla^2 W(t)|) \tag{57}
$$

$$
|D_p Q| \le \mu(|W(t)|/r + |\nabla W(t)|) \tag{58}
$$

Using properties of Q above together with estimate i, ii, iii, the sup norm of $\dot{F}$ in A_r can then be estimated as

$$
\left|\frac{d}{dt}F(W(t))\right| \le C r^{2\nu-3} ||W(t)||_{\mathcal{B}} ||\dot{W}||_{\mathcal{B}} \tag{59}
$$

$$
\le C r^{2\nu-3} \left(||W_1||_{\mathcal{B}} + ||W_2||_{\mathcal{B}}\right) ||W_1 - W_2||_{\mathcal{B}} \tag{60}
$$

Similarly, the Holder norm of $\dot{F}$ in A_r can also be estimated using estimate iv, v, vi and properties of Q:

$$
\left[\frac{d}{dt}F(W(t))\right]_\alpha \le C r^{2\nu-3-\alpha} ||W(t)||_{\mathcal{B}} ||\dot{W}||_{\mathcal{B}}
$$

Combining both estimate and argue similarly as in the sup norm case, we obtain the second inequality

$$
||F(W_1) - F(W_2)||_{C^{0,\alpha}_{\nu-2}} \le C(||W_1||_{\mathcal{B}} + ||W_2||_{\mathcal{B}}) ||W_1 - W_2||_{\mathcal{B}}
$$

□

Now we shall apply Lemma 11 to check the properties of the operator $\mathcal{U}$. The first inequality in Lemma 11 and Theorem 9 tell us that the image $\mathcal{U}(W) \in \mathcal{B}$ whenever $W \in \mathcal{B}$. So we have

$$\mathcal{U} : \mathcal{B} \to \mathcal{B}$$

The next step is to suitably choose the constant A such that $\mathcal{U}(W) \in \mathcal{K}$ whenever $W \in \mathcal{K}$. By Theorem 9, for any $W \in \mathcal{K}$, the solution $\mathcal{U}(W)$ satisfies

$$\begin{aligned} ||\mathcal{U}(W)||_{\mathcal{B}} &\leq K_1 \left(||F(W)||_{C^{0,\alpha}_{\nu-2}} + ||\Pi_J H||_{C^{2,\alpha}(M)} \right) \\ &\leq K_1 \left(K_2 ||W||^2_{\mathcal{B}} + ||\Pi_J H||_{C^{2,\alpha}} \right) && \text{(by Lemma 11)} \\ &\leq K_1 \left(K_2 A^2 ||\Pi_J H||^2_{C^{2,\alpha}} + ||\Pi_J H||_{C^{2,\alpha}} \right) && \text{(by properties of } \mathcal{K}) \end{aligned}$$

If $||\Pi_J H||_{C^{2,\alpha}} < 1$, we want to find A satisfying the following inequality

$$||\mathcal{U}(W)||_{\mathcal{B}} \leq K_1 \left(K_2 A^2 ||\Pi_J H||^2_{C^{2,\alpha}} + ||\Pi_J H||_{C^{2,\alpha}} \right) \leq A ||\Pi_J H||_{C^{2,\alpha}}.$$

This is equivalent to solving the quadratic inequality of A, that is

$$K_1 K_2 A^2 ||\Pi_J H||_{C^{2,\alpha}} - A + K_1 \leq 0$$

If we let $A = 2K_1$, then the inequality above is true as long as

$$||\Pi_J H||_{C^{2,\alpha}} \leq \frac{1}{4K_1^2 K_2}$$

The final step is to show that $\mathcal{U}$ is a contraction mapping in $\mathcal{K}$. Let $\Phi = \mathcal{U}(W_1) - \mathcal{U}(W_2)$, where $W_1, W_2 \in \mathcal{K}$, then Φ satisfies the boundary value problem

$$L\Phi = F(W_1) - F(W_2) \quad \text{on } C_1^* \tag{61}$$

$$\Pi_J \Phi = 0 \quad \text{on } M \tag{62}$$

Applying Schauder estimate from Proposition 3 and the second inequality in Lemma 11, we have

$$||\mathcal{U}(W_1) - \mathcal{U}(W_2)||_{\mathcal{B}} = ||\Phi||_{\mathcal{B}} \tag{63}$$

$$\leq c||F(W_1) - F(W_2)||_{C^{0,\alpha}_{\nu-2}} \tag{64}$$

$$\leq c'(||W_1||_{\mathcal{B}} + ||W_2||_{\mathcal{B}})||W_1 - W_2||_{\mathcal{B}} \tag{65}$$

$$\leq 2Ac' \|\Pi_J H\|_{C^{2,\alpha}} ||W_1 - W_2||_{\mathcal{B}} \tag{66}$$

$$= 4K_1 c' \|\Pi_J H\|_{C^{2,\alpha}} ||W_1 - W_2||_{\mathcal{B}} \tag{67}$$

Now if $\|\Pi_J H\|_{C^{2,\alpha}} < \epsilon_0 := \min\{1, 1/(4K_1K_2^2), 1/(4K_1c')\}$ then $\mathcal{U}$ is a contraction mapping from $\mathcal{K}$ to itself. The contraction mapping principle then says that there exist a unique fix point $\Phi \in \mathcal{K}$ for $\mathcal{U}$ i.e. $L\Phi = F(\Phi) = -Q(\Phi/r, \nabla^\perp\Phi, (\nabla^\perp)^2\Phi)$, hence Φ solves $M_G(\Phi) = 0$. Moreover it satisfies the estimate

$$||\Phi||_{\mathcal{B}} \leq A||\Pi_J H||_{C^{2,\alpha}}$$

We summarize the result as follow

Theorem 12 *Let M^{n-1} be a smooth minimal submanifold in $\mathbb{S}^{n+k-1}$ and C be cone over M. Given $\alpha \in (0, 1)$, $\nu > 3$ and $J \geq 1$ such that $\Re\gamma_J < \nu < \Re\gamma_{J+1}$, then there exist positive constants A and ϵ_0 (depending only M, n, k, ν, α) such that corresponding to any $H \in C^{2,\alpha}(M)$ with $||\Pi_J H||_{C^{2,\alpha}(M)} < \epsilon_0$ there exist a solution $\Phi \in C^{2,\alpha}_\nu(C^*_1; NC)$ to the zero mean curvature operator equation (2), i.e.*

$$M_G(\Phi) = 0 \text{ on } C^*_1,$$

with boundary condition

$$\Pi_J\Phi = \Pi_J H \text{ on } M$$

Moreover, such Φ also satisfies the estimate

$$||\Phi||_{C^{2,\alpha}_\nu(C^*_1)} \leq A||\Pi_J H||_{C^{2,\alpha}(M)}$$

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